CMS/CAIMS Books in Mathematics

Canadian
Mathematical Society
Société mathématique
du Canada

Arian Novruzi

A Short Introduction to Partial Differential Equations

CAIMS
SCMAI

Springer

CMS/CAIMS Books in Mathematics

Volume 11

CMS/CAIMS Books in Mathematics is a collection of monographs and graduate-level textbooks published in cooperation jointly with the Canadian Mathematical Society- Societé mathématique du Canada and the Canadian Applied and Industrial Mathematics Society-Societé Canadienne de Mathématiques Appliquées et Industrielles. This series offers authors the joint advantage of publishing with two major mathematical societies and with a leading academic publishing company. The series is edited by Karl Dilcher, Frithjof Lutscher, Nilima Nigam, and Keith Taylor. The series publishes high-impact works across the breadth of mathematics and its applications. Books in this series will appeal to all mathematicians, students and established researchers. The series replaces the CMS Books in Mathematics series that successfully published over 45 volumes in 20 years.

CMS
SMC

CAIMS
SCMAI

Arian Novruzi

A Short Introduction to Partial Differential Equations

Springer

Arian Novruzi
Department of Mathematics
University of Ottawa
Ottawa, ON, Canada

ISSN 2730-650X ISSN 2730-6518 (electronic)
CMS/CAIMS Books in Mathematics
ISBN 978-3-031-39523-9 ISBN 978-3-031-39524-6 (eBook)
https://doi.org/10.1007/978-3-031-39524-6

Mathematics Subject Classification: 35FXX, 65M25, 35BXX, 46FXX, 46EXX, 35JXX, 35KXX, 35LXX

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Paper in this product is recyclable.

To my beloved parents, Xhiko and Milo

Preface

There are many publications on Partial Differential Equations (PDEs) in a multitude of forms, varying from research-oriented papers and books to graduate/undergraduate textbooks. The number of topics and the depth of analysis in these publications also varies. Some of them address a limited number of topics, while others cover a broader spectrum of topics, with a level of difficulty varying from an introductory to an advanced or expert level.

This book provides a short introduction to PDEs. It is primarily addressed to graduate students and researchers, who are new to PDEs. The book offers a user-friendly approach to the analysis of PDEs, by combining elementary techniques with important concepts and fundamental modern methods.

This book focuses on the analysis of four prototypes of PDEs: first-order PDEs, second-order linear elliptic PDEs, and two linear evolution PDEs—heat and wave equations, each with a second-order linear elliptic PDE principal term. To facilitate a smooth introduction into the analysis for the reader, two approaches are presented for each of the PDEs. The first approach consists of the method of analytical and classical solutions. It includes the method of characteristics, the method of separation of variables, and Perron's method. The simplicity of this approach, its potential to provide classical solutions, and the impossibility of providing a classical solution in a general setting are highlighted, and used to motivate the second approach.

The second approach is the method of weak (variational) solutions. While for the first-order PDEs we focus the analysis on a short discussion of scalar conservation laws, we present a detailed analysis for the other PDEs. The main ingredients we use are the Lax-Milgram lemma, Fredholm alternative and spectral decomposition theorems, and a number of results from Sobolev spaces. As an application of the results from the second-order linear elliptic PDEs and Sobolev spaces, we give a very short introduction to the solution to certain nonlinear PDEs by using a fixed point approach.

In connection with the second approach, we give an introduction to distributions, Fourier transform, and Sobolev spaces, which are fundamental for the study of PDEs. Though in a number of cases we deal with general fractional Sobolev spaces $W^{s,p}$, we mainly analyze H^s Sobolev spaces by using Fourier transform. We give some fundamental results related to the concepts of density, extension, embeddings, compactness, and boundary traces. We provide proofs of some of these results in particular cases, with the intention to highlight the most relevant techniques and to avoid the complexity of general cases. The book ends with an appendix chapter, which complements the previous chapters with proofs, examples, and remarks.

Chapter 1 (introduction) provides some concepts and results from analysis, functional analysis, and topology, which will be used as the book develops. At first reading, one only needs to get familiar with them. Chapter 2 provides the formal definition of PDEs and a number of examples. Chapters 3 (first-order PDEs) and 4 (second-order PDEs and maximum principle) can be read independently. Each of them depends on certain results from chapter 1. Chapters 7 (second-order PDEs and weak solutions) and 8 (evolution PDEs) are almost independent, and each of them depends on chapters 6 (Sobolev spaces) and 5 (distributions), and some results from chapter 1.

This book can be used for an intense one-semester, or normal two-semester, PDE course. The reader is expected to have knowledge of linear algebra and differential equations, a good background in real and complex calculus, and a modest background in analysis and topology. The book has many examples, which help to explain the concepts, highlight the key ideas, and emphasize the sharpness of results, as well as a section of problems at the end of each chapter.

I am grateful to professor Michel Pierre for a number of comments and suggestions, which have been helpful for me to improve this book.

Ottawa, Canada
June 2023

Arian Novruzi

Contents

1. Notations and review

The objective of this chapter is to help the reader become familiar with a number of concepts and some important results that will be used throughout this book. It is expected that the reader has a minimum background in analysis in $\mathbb{R}^N$ and topology.

1.1 Continuous differentiable functions

In this book, $\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ the set of complex numbers, and for $N \in \mathbb{N}$, $\mathbb{R}^N = \{x = (x_1, \ldots, x_N),\ x_1, \ldots, x_N \in \mathbb{R}\}$, $\mathbb{C}^N = \{x = (x_1, \ldots, x_N),\ x_1, \ldots, x_N \in \mathbb{C}\}$.

Before we start with the definition of classical continuous differentiable functions in Banach spaces, we are reminded the simplest of Banach spaces $(\mathbb{R}, |\cdot|)$, with $|\cdot|$ the absolute value norm. Another Banach space is $(\mathbb{R}^N, |\cdot|_p)$, where $N \in \mathbb{N}$, $p \in [1, \infty]$ and $|\cdot|_p$ is given by

$$\forall x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \quad |x|_p = \begin{cases} (|x_1|^p + \cdots + |x_N|^p)^{1/p}, & p \in [1, \infty), \\ \max\{|x_1|, \ldots, |x_N|\}, & p = \infty. \end{cases} \tag{1.1.1}$$

In general, a Banach space is defined as follows.

Definition 1.1.1 (Banach spaces) *A Banach space is a couple $(X, \|\cdot\|_X)$, where X is a vector space over the field of real numbers $\mathbb{R}$ (or of complex numbers $\mathbb{C}$) and $\|\cdot\|_X$ is a norm, such that X is complete with respect to the norm $\|\cdot\|_X$, i.e.*[1] *every Cauchy*[2] *sequence (x_n) in X converges to an element $x \in X$ in the norm $\|\cdot\|_X$, which is equivalent to*

$$\lim_{n\to\infty} \|x_n - x\|_X = 0.$$

We note that if $(X_i, \|\cdot\|_{X_i})$ are Banach spaces, $Z = \prod_{i=1}^n X_i = X_1 \times \cdots \times X_n$ and

$$\|z\|_Z = \max\{\|x_1\|_{X_1}, \ldots, \|x_n\|_{X_n}\}, \quad (or \ \ \|z\|_Z = \|x_1\|_{X_1} + \cdots + \|x_n\|_{X_n}) \tag{1.1.2}$$

[1] In Latin *id est*, which means *that is*

[2] (x_n) is a Cauchy sequence if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $\|x_{n+m} - x_n\|_X < \epsilon$ for all integers $n > n_\epsilon$ and $m \in \mathbb{N}$.

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6_1

for $z = (x_1, \dots, x_n) \in Z$, then it is easy to show that $(Z, \|\cdot\|_Z)$ is another Banach space. If $X_1 = \cdots = X_n = X$ we write X^n instead of $\underbrace{X \times \cdots \times X}_{n \text{ times}}$.

Definition 1.1.2 ($C^0(\Omega; Y)$ spaces) *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces, $\Omega \subset X$ be an open set, and $u : \Omega \mapsto Y$ a function. We say "u is continuous at $x \in \Omega$" if*

$$\lim_{\|h\|_X \to 0} \|u(x+h) - u(x)\|_Y = 0 \tag{1.1.3a}$$

$$\Updownarrow$$

$$\forall \epsilon > 0, \ \ \exists \delta_{\epsilon,x} > 0, \ \ \forall h \in \Omega, \ \ \|h\|_X < \delta_\epsilon \Rightarrow \|u(x+h) - u(x)\|_Y < \epsilon. \tag{1.1.3b}$$

If u is continuous at every $x \in \Omega$, we say "u is continuous in Ω". The set of such functions u is denoted by $C^0(\Omega; Y)$, or simply $C^0(\Omega)$ if $Y = \mathbb{R}$.

A function $u \in C^0(\Omega; Y)$ is said to be "uniformly continuous in Ω" if there exists $\delta_\epsilon > 0$ such that $\delta_{\epsilon,x}$ in (1.1.3b) satisfies $\delta_{\epsilon,x} \geq \delta_\epsilon$ for all $x \in \Omega$.

We will see below that the definition of the derivatives of functions involve the spaces of linear continuous functions, which are simple examples of Banach spaces.

Definition 1.1.3 ($\mathcal{L}(X^n; Y)$ spaces) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces and $n \in \mathbb{N}$.*

i) $\ell : X^n \mapsto Y$ is said to be "n-linear" if

$$\begin{aligned} \ell(x_1, \dots, x_{i-1}, c_i' x_i' + c_i'' x_i'', x_{i+1}, \dots, x_n) &= c_i' \ell(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n) \\ &+ c_i'' \ell(x_1, \dots, x_{i-1}, x_i'', x_{i+1}, \dots, x_n), \end{aligned}$$

for all $i = 1, \dots, n$, $x_1, \dots, x_{i-1}, x_i', x_i'', x_{i+1}, \cdots, x_n \in X$ and $c_i', c_i'' \in \mathbb{R}$.

ii) $\mathcal{L}(X^n; Y)$, or simply $\mathcal{L}(X; Y)$ when $n = 1$, is defined as

$$\mathcal{L}(X^n; Y) = \{\ell \in C^0(X^n; Y), \ \ \ell \ \text{ is } n\text{-linear}\}. \tag{1.1.4}$$

Given $\ell \in \mathcal{L}(X^n; Y)$ and $x \in X$, sometimes instead of $\ell(x)$ we will write $\langle \ell, x \rangle_{\mathcal{L}(X^n;Y) \times X^n}$, and we will remove $\mathcal{L}(X^n; Y) \times X^n$ from $\langle \ell, x \rangle$ whenever there is no ambiguity. Also, we will omit Y when $Y = \mathbb{R}$, so we will write $\mathcal{L}(X^n)$ instead of $\mathcal{L}(X^n; \mathbb{R})$, and furthermore we will write X', resp. X'' instead of $\mathcal{L}(X)$, resp. $\mathcal{L}(X'')$.

Remark 1.1.4 *A n-linear map ℓ from X^n in Y is continuous if and only if*

$$\|\ell(x)\|_Y \leq C \|x\|_{X^n}, \quad \forall x \in X^n,$$

with a certain $C \geq 0$ independent of x, see [26], and

$$\|\ell(x)\|_Y = \inf\{C \geq 0, \ \ \|\ell(x)\|_Y \leq C \|x\|_{X^n}, \quad \forall x \in X^n\}. \tag{1.1.5}$$

For this reason, a "continuous n-linear map" is equivalent to a "bounded n-linear map". Furthermore, one can show that

$$\|\ell\|_{\mathcal{L}(X^n;Y)} = \sup\{\|\ell(x)\|_Y,\ x \in X^n,\ \|x\|_{X^n} = 1\}, \quad \ell \in \mathcal{L}(X^n;Y), \tag{1.1.6}$$

is a norm in $\mathcal{L}(X^n;Y)$ and $(\mathcal{L}(X^n;Y), \|\cdot\|_{\mathcal{L}(X^n;Y)})$ is a Banach space.

Example 1.1.5 *In the case when $X = \mathbb{R}^N$ and $Y = \mathbb{R}$ we have*

$$\begin{aligned}
\mathcal{L}(\mathbb{R}^N) &= \Big\{\ell : \mathbb{R}^N \mapsto \mathbb{R}, \quad \ell(x) = \sum_{i=1}^{N} A_i x_i, \\
&\qquad A = (A_i) \in \mathbb{R}^N, \quad x = (x_i) \in \mathbb{R}^N\Big\}, \\
\mathcal{L}(\mathbb{R}^{2N}) &= \Big\{\ell : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}, \quad \ell(x^1, x^2) = \sum_{i,j=1}^{N} A_{i,j} x_i^1 x_j^2, \\
&\qquad A = (A_{i,j}) \in \mathbb{R}^{N^2}, \quad x^1 = (x_i^1), \ \ x^2 = (x_j^2) \in \mathbb{R}^N\Big\}, \\
\cdots &: \cdots
\end{aligned}$$

Now we define the C^k spaces in general Banach spaces.

Definition 1.1.6 ($C^k(\Omega;Y)$ spaces) *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces, $\Omega \subset X$ be an open set, and $u \in C^0(\Omega;Y)$.*

1) We say u is differentiable at $x \in \Omega$ if there exists $u^{(1)}(x) \in \mathcal{L}(X;Y)$, called "derivative of u at x", such that

$$\lim_{\|h\|_X \to 0} \frac{\|u(x+h) - u(x) - u^{(1)}(x;h)\|_Y}{\|h\|_X} = 0.$$

Here $u^{(1)}(x;h)$ denotes the value of $u^{(1)}(x)$ at h. If u is differentiable at every $x \in \Omega$, then $u^{(1)} : \Omega \mapsto \mathcal{L}(X;Y)$ denotes "the embedded derivative of u". Furthermore we set

$$C^1(\Omega;Y) = \{u \in C^0(\Omega;Y) \ \text{ such that } \ u^{(1)} \in C^0(\Omega; \mathcal{L}(X;Y))\}. \tag{1.1.7}$$

2) For $k = 2, 3, \ldots$, "the k-th order derivative of u at x" is defined by $u^{(k)}(x) := (u^{(k-1)})^{(1)}(x) \in \underbrace{\mathcal{L}(X; \mathcal{L}(X; \ldots, \mathcal{L}(X; \mathcal{L}(X;Y))))}_{k \text{ times}}$, and if it is continuous in Ω set

$$\begin{aligned}
C^k(\Omega;Y) &= \{u \in C^0(\Omega;Y) \ \text{ such that for all} n = 1, \ldots, k \text{ we have} \\
&\qquad u^{(n)} \in C^0(\Omega; \underbrace{\mathcal{L}(X; \mathcal{L}(X; \mathcal{L}(X; \ldots, \mathcal{L}(X; \mathcal{L}(X;Y))))))}_{n \text{ times}}\}.
\end{aligned} \tag{1.1.8}$$

$$\text{We also define} \quad C^\infty(\Omega;Y) = \bigcap_{k \in \mathbb{N}} C^k(\Omega;Y). \tag{1.1.9}$$

3) The space of Lipschitz functions is defined by

$$\begin{aligned} Lip(\Omega;Y) &= C^{0,1}(\Omega;Y) \\ &= \{u \in C^0(\Omega;Y),\ \exists L = L(u) \in [0,\infty) \text{ such that} \\ &\quad \forall x,y \in \Omega,\ \|u(x)-u(y)\|_Y \leq L\|x-y\|_X\}. \end{aligned} \tag{1.1.10}$$

4) When $Y = \mathbb{R}$ we will omit Y in the spaces of this definition, so we will write $C^1(\Omega)$, $C^k(\Omega)$, $C^\infty(\Omega)$, $Lip(\Omega)$. Also, we will write u', resp. u'', u''', instead of $u^{(1)}$, resp. $u^{(2)}$, $u^{(3)}$.

Remark 1.1.7 *Instead of the spaces $\underbrace{\mathcal{L}(X;\mathcal{L}(X;\ldots,\mathcal{L}(X;\mathcal{L}(X;Y))))}_{k \text{ times}}$ in Definition 1.1.6 we can use $\mathcal{L}(X^k;Y)$. This follows from the fact that there exists a linear bijective isometry $i : \mathcal{L}(X^n;\mathcal{L}(X^m,Y)) \mapsto \mathcal{L}(X^{n+m};Y)$ defined by for $\ell_{n,m} \in \mathcal{L}(X^n;\mathcal{L}(X^m,Y))$,*

$$i(\ell_{n,m})(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+m}) = \ell_{n,m}(x_1,\ldots,x_n)(x_{n+1},\ldots,x_{n+m}). \tag{1.1.11}$$

The proof is simple and we refer the reader to [8].

Example 1.1.8 *Let $a,b \in \mathbb{R}$, $a < b$, $X = C^0([a,b])$ be the set of continuous functions from $[a,b]$ to $\mathbb{R}$ endowed with the maximum norm, $\|u\|_X = \max\{|u(x)|,\ x \in [a,b]\}$. It is a classical fact that $(X,\|\cdot\|_X)$ is a Banach space; see [7, 26]. Let $f : X \mapsto X$, $f(u) = \dfrac{1}{1+u^2}$. Clearly $f(u) \in X$ and $f \in C^0(X;X)$ because*

$$\begin{aligned} \|f(u+h)-f(u)\|_X &= \left\| \frac{1}{1+(u+h)^2} - \frac{1}{1+u^2} \right\|_X \leq (\|h\|_X + 2\|u\|_X)\|h\|_X \\ &\to 0 \text{ as } \|h\|_X \to 0. \end{aligned}$$

Furthermore, $f'(u)$ exists for all $u \in X$ and $f'(u;h) = -2\dfrac{uh}{(1+u^2)^2}$. Indeed, clearly $h \mapsto f'(u;h) \in \mathcal{L}(X;X)$ because for $h \in X$ we have $\|f'(u;h)\|_X \leq 2\|u\|_X\|h\|_X$. Furthermore

$$\begin{aligned} \frac{1}{\|h\|_X}\|f(u+h)-f(u)-f'(u;h)\|_X &\leq (1+2\|h\|_X\|u\|_X + 3\|u\|_X^2)\|h\|_X \\ &\to 0 \text{ as } \|h\|_X \to 0. \end{aligned}$$

Finally, $f' \in C^0(X;\mathcal{L}(X;X))$ because

$$\begin{aligned} \|f'(u+v)-f'(u)\|_{\mathcal{L}(X;X)} &= \sup\{\|f'(u+v;h)-f'(u;h)\|_X,\ \|h\| \leq 1\} \\ &= 2\sup\left\{ \left\| \frac{(u+v)h}{(1+(u+v)^2)^2} - \frac{uh}{(1+u^2)^2} \right\|_X,\ \|h\| \leq 1 \right\} \\ &\leq 2\left\| \frac{(u+v)}{(1+(u+v)^2)^2} - \frac{u}{(1+u^2)^2} \right\|_X \\ &\to 0 \text{ as } \|v\|_X \to 0. \end{aligned}$$

Actually, one can show that $f \in C^\infty(X;Y)$.

In the following we consider $X = \mathbb{R}^N$ and $Y = \mathbb{R}$. In this case we have equivalent definitions of $C^k(\Omega)$ spaces, which are given in terms of partial derivatives with which the reader is familiar. But first, let us introduce the following notations that will be used throughout this book.

Notations 1.1.9 *A $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$ is called a "multi-index". For $\alpha, \beta \in \mathbb{N}_0^N$, $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, and $m \in \mathbb{N}$, we define*

$$\begin{cases} |\alpha| = \alpha_1 + \cdots + \alpha_N, & \alpha \leq \beta \Longleftrightarrow \alpha_i \leq \beta_i, \ \forall i, \\ \alpha! = \alpha_1! \cdots \alpha_N!, & \alpha \leq m \Longleftrightarrow \alpha_i \leq m, \ \forall i, \\ x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}. \end{cases} \tag{1.1.12}$$

For $\alpha \in \mathbb{N}_0^N$ and $u : \mathbb{R}^N \mapsto \mathbb{R}$ having continuous partial derivatives of order $|\alpha|$ in the usual sense, we write

$$\begin{aligned} D^\alpha u &= \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N} u = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N} u = u_{\underbrace{x_1 \cdots x_1}_{\alpha_1 \ times} \cdots\cdots \underbrace{x_N \cdots x_N}_{\alpha_N \ times}}, \\ D^\alpha u &= u, \quad if\ |\alpha| = 0. \end{aligned} \tag{1.1.13}$$

We say $D^\alpha u$ is an "α derivative of u", and a "$D^\alpha u$ is a derivative of order $|\alpha|$". Finally, for $m \in \mathbb{N}_0$ we denote by $D^m u = (u_1, u_2, \ldots, u_{N^m})$ the vector of m-th order partial derivatives of u, defined by recurrence as follows:

$$\begin{aligned} &D^0 u := u, \\ &D^1 u = (D_1^1, \ldots, D_N^1) := (\partial_1 u, \ldots, \partial_N u), \\ &D^2 u = (D_1^2, \ldots, D_{N^2}^2) := (D^1 D_1^1 u, \ldots, D^1 D_N^1 u), \\ &\quad \cdots \\ &D^m u = (D_1^m, \ldots, D_{N^m}^m) := (D^1 D_1^{m-1} u, \ldots, D^1 D_{N^{m-1}}^{m-1} u). \end{aligned}$$

Often we write Du or ∇u instead of $D^1 u$.

Proposition 1.1.10 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then the spaces $C^k(\Omega)$ of Definition 1.1.6 with $X = \mathbb{R}^N$, $Y = \mathbb{R}$, are equivalently defined as*

$$C^k(\Omega) = \{u : \Omega \mapsto \mathbb{R}, \ D^\alpha u \in C^0(\Omega), \ \forall \alpha \in \mathbb{N}_0^N, \ |\alpha| \leq k\}, \tag{1.1.15}$$

For the proof the reader can see [7, 8]. The idea of the proof, in the case $k = 1$ to avoid the technicalities, is to point out that if $u \in C^1(\Omega)$ in the sense of Definition 1.1.6, then necessarily $u \in C^0(\Omega)$ and all $\partial_{x_i} u(x) := u'(x; e_i)$, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the i-th place, exist and are continuous in Ω. Conversely, if all $\partial_{x_i} u \in C^0(\Omega)$ then one defines $u'(x) \in \mathcal{L}(\mathbb{R}^N)$ by $u'(x; h) = \sum_{i=1}^n \partial_{x_i} u(x) h_i$, $h = (h_1, \ldots, h_n)$, and proves that $u \in C^1(\Omega)$ in the sense of Definition 1.1.6 by using the mean value theorem and the continuity of all $\partial_{x_i} u$.

In general, $C^k(\Omega)$ is not a Banach space because it contains unbounded functions. However, its subspace of bounded functions is a Banach space.

Proposition 1.1.11 *Let $\Omega \subset \mathbb{R}^N$ be open, $k \in \mathbb{N}_0$, $\lambda \in (0,1]$, and define*

$$C_b^0(\Omega) = \{u \in C^0(\Omega),\ u \text{ is bounded in } \Omega\}, \tag{1.1.16}$$
$$C_b^k(\Omega) = \{u \in C_b^0(\Omega),\ D^\alpha u \in C_b^0(\Omega),\ \forall \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| \le k\}, \tag{1.1.17}$$
$$C_b^{k,\lambda}(\Omega) = \left\{ u \in C_b^k(\Omega),\ |u|_{C_b^{k,\lambda}(\Omega)} := \sup_{\substack{|\alpha|=k \\ \Omega \ni x \neq y \in \Omega}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda} < \infty \right\}, \tag{1.1.18}$$
$$C_b^\infty(\Omega) = \bigcap_{k \in \mathbb{N}_0} C_b^k(\Omega). \tag{1.1.19}$$

The space $C_b^k(\Omega)$, resp. $C_b^{k,\lambda}(\Omega)$, $k \in \mathbb{N}_0$, equipped with the norm

$$\|u\|_{C_b^k(\Omega)} = \sup\{\|D^\alpha u\|_{C_b^0(\Omega)},\ |\alpha| \le k\}, \quad \text{resp.} \tag{1.1.20}$$
$$\|u\|_{C_b^{k,\lambda}(\Omega)} = \|u\|_{C_b^k(\Omega)} + |u|_{C_b^{k,\lambda}(\Omega)}, \tag{1.1.21}$$

is a Banach space.

Proof. See Proposition 9.1.1 in Annex. □

The spaces $C^{k,\lambda}$ are called "*Hölder spaces*". They are useful when describing additional regularity of functions in Sobolev spaces; see chapter 6. We note that in connection with $C^{k,\lambda}$ spaces there is a complete theory associated with second-order linear partial differential equations, called "*Hölder theory*"; see, for example, [20].

We also note that a function in $C_b^k(\Omega)$ is not necessarily continuous up to the boundary. For example, $u(x) = \dfrac{2x_1x_2}{x_1^2 + x_2^2} \in C_b^0(\Omega)$, $\Omega = (0,1) \times (0,1)$, but $u \notin C_b^0(\overline{\Omega})$. The following subspaces of $C_b^0(\Omega)$ are useful when approximating, or describing additional regularity, of Sobolev spaces functions, and are defined by

$$C_b^0(\overline{\Omega}) = \{u \in C_b^0(\Omega),\ u \text{ extends continuously in } \overline{\Omega}\}, \tag{1.1.22}$$
$$C_b^k(\overline{\Omega}) = \{u \in C_b^0(\overline{\Omega}),\ D^\alpha u \in C_b^0(\overline{\Omega}),\ \forall \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| \le k\}, \tag{1.1.23}$$
$$C_b^{k,\lambda}(\overline{\Omega}) = \left\{ u \in C_b^k(\overline{\Omega}),\ |u|_{C^{k,\lambda}(\overline{\Omega})} := \sup_{\substack{|\alpha|=k \\ \overline{\Omega} \ni x \neq y \in \overline{\Omega}}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda} < \infty \right\},$$
$$C_b^\infty(\overline{\Omega}) = \bigcap_{k \in \mathbb{N}_0} C_b^k(\overline{\Omega}), \tag{1.1.24}$$

where $\overline{\Omega}$ is the closure of Ω.[3,4] If Ω is bounded, we can remove the subscript "b" from the spaces $C_b^k(\overline{\Omega})$ and $C_b^{k,\lambda}(\overline{\Omega})$, because every continuous function in a compact set[5] is bounded. The space $C_b^k(\overline{\Omega})$, resp. $C_b^{k,\lambda}(\overline{\Omega})$, endowed with the norm (1.1.20), resp. (1.1.21), is a Banach space.

When working with the distributions or the weak solution to PDEs, we are interested in $C^k(\Omega)$ functions which are zero in a neighborhood of the boundary $\partial\Omega$ of Ω.

Definition 1.1.12 ($C_0^k(\Omega)$ spaces) *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then we define*

$$\begin{aligned} C_0^0(\Omega) &= \{u \in C^0(\Omega), \\ &\qquad \mathrm{supp}(u) := \overline{\{x \in \Omega,\ u(x) \neq 0\}} \text{ is compact and included in } \Omega\}, && (1.1.25)\\ C_0^k(\Omega) &= \{u \in C^k(\Omega),\ \ D^\alpha u \in C_0^0(\Omega),\ \ \alpha \in \mathbb{N}^N,\ |\alpha| \le k\}, && (1.1.26)\\ C_0^\infty(\Omega) &= \bigcap_{k \in \mathbb{N}\cup\{0\}} C_0^k(\Omega) =: \mathcal{D}(\Omega). && (1.1.27)\end{aligned}$$

Example 1.1.13 *Let $\rho : \mathbb{R}^N \mapsto \mathbb{R}$ be defined by*

$$\rho(x) = \frac{1}{C}\begin{cases} e^{\frac{1}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases} \qquad C = \int_{\{|x|<1\}} e^{\frac{1}{|x|^2-1}} dx. \tag{1.1.28}$$

It is easy to verify that

$$i)\ \rho \in \mathcal{D}(\mathbb{R}^N), \quad supp(\rho)\{|x| \le 1\}, \qquad ii) \int_{\mathbb{R}^N} \rho(x)dx = 1. \tag{1.1.29}$$

We set $\rho_n(x) = n^N \rho(nx)$, $n \in \mathbb{N}$. The sequence (ρ_n) satisfies

$$\begin{cases} \rho_n \in \mathcal{D}(\mathbb{R}^N), \quad \rho_n \ge 0, \\ \mathrm{supp}(\rho_n) \subset \left\{x \in \mathbb{R}^N,\ |x| \le \frac{1}{n}\right\}, \\ \int_{\mathbb{R}^N} \rho_n(x)dx = 1. \end{cases} \tag{1.1.30}$$

Every function satisfying (1.1.29) is called a "mollifier". Given a mollifier ρ, the sequence (ρ_n), which necessarily satisfies (1.1.30), is called a "mollifier sequence". The mollifier sequences are useful when used with the convolution operator; see Theorem 1.3.5.

[3]The closure $\overline{\Omega}$ of $\Omega \subset X$, X Banach, is defined as $\overline{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$ is the boundary of Ω.

[4]The boundary $\partial\Omega$ of $\Omega \subset X$, X Banach, is defined as $\partial\Omega = \{x \in X,\ \ B(x,r) \cap \Omega \neq \emptyset,\ \ B(x,r) \cap \Omega^c \neq \emptyset,\ \ \forall r > 0\}$.

[5]In general, $K \subset X$, X Banach, is compact if every sequence (x_n) in K has a convergent sequence with limit in K.

Usually, PDEs are equipped with boundary conditions, which describe the solution on the boundary. It is then necessary to define some spaces of functions on the boundary. First we define the space $C^0(\partial\Omega)$. The definition of $C^k(\partial\Omega)$ spaces requires more regularity of $\partial\Omega$ and will be introduced in the next section.

Definition 1.1.14 ($C^0(\partial\Omega)$ spaces) *Let $\Omega \subset \mathbb{R}^N$. A function $u : \partial\Omega \mapsto \mathbb{R}$ is said to be "continuous at $x \in \partial\Omega$" if $\lim\limits_{\substack{|y-x|\to 0\\ y\in\partial\Omega}} |u(y) - u(x)| = 0$, and it is said to be "continuous on $\partial\Omega$" if it is continuous at every $x \in \partial\Omega$. We define $C^0(\partial\Omega)$ by*

$$C^0(\partial\Omega) \quad = \quad \{u : \partial\Omega \mapsto \mathbb{R},\ u \text{ continuous on } \partial\Omega\}. \tag{1.1.31}$$

When $\partial\Omega$ is unbounded one defines $C^0_b(\partial\Omega) = C^0(\partial\Omega) \cap \{u \text{ bounded}\}$.

We note that when Ω is bounded, we have $C^0_b(\partial\Omega) = C^0(\partial\Omega)$. The space $C^0_b(\partial\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{C^0(\partial\Omega)} \quad = \quad \max\{|u(x)|,\ x \in \partial\Omega\}. \tag{1.1.32}$$

1.2 Domains and $C^k(\partial\Omega)$ spaces

In this section we will introduce the concept of regularity of domains, which is necessary for the definition of classical $C^k(\partial\Omega)$ spaces (and also boundary Sobolev spaces—we will see this in chapter 6). Regularity is an important property of the domain and it strongly affects the regularity to the solution to partial differential equations near the boundary; see section 9.6.

1.2.1 Partition of unity

Many proofs of PDE results involve a reduction to local considerations around a point, by using the so-called "*partition of unity*".

Definition 1.2.1 (Partition of unity) *Let $\Gamma \subset \mathbb{R}^N$ be a compact set and $\{G_1, \dots, G_m\}$ be a family of open bounded sets in $\mathbb{R}^N$ covering Γ, i.e. $\Gamma \subset \bigcup_{i=1}^m G_i$. A partition of unity subordinate to the covering $\{G_1, \dots, G_m\}$ of Γ is $\{\varphi_0, \varphi_1, \dots, \varphi_m\}$, with $\varphi_0 \in C^\infty(\mathbb{R}^N)$ and $\varphi_1, \dots, \varphi_m \in \mathcal{D}(\mathbb{R}^N)$ satisfying*

i) $0 \le \varphi_i \le 1$, *for all* $i = 0, 1, \dots, m$,

ii) $\sum\limits_{i=0}^{m} \varphi_i(x) = 1$ *for all* $x \in \mathbb{R}^N$,

iii) $\mathrm{supp}(\varphi_i) \subset G_i$ *for all* $i = 1, \dots, m$, $\ \mathrm{supp}(\varphi_0) \cap \Gamma = \emptyset$.

Note that in the case $\Gamma = \partial\Omega$ with $\Omega \subset \mathbb{R}^N$ an open bounded set, we have $\theta_0 = 0$ in a neighborhood of $\partial\Omega$, and so $\sum_{i=1}^m \varphi_i = 1$ in the same neighborhood of $\partial\Omega$.

Theorem 1.2.2 (cut-off function and partition of unity)

i) **(cut-off function)** *Let $K \subset \mathbb{R}^N$ be compact and $G \subset \mathbb{R}^N$ open with $K \subset G$. There exists $\eta \in \mathcal{D}(\mathbb{R}^N)$ such that*

$$0 \le \eta \le 1, \quad \eta = 1 \ \textit{in} \ K \ \textit{and} \ \mathrm{supp}(\eta) \subset G. \tag{1.2.2}$$

Such a function η is called a "$\{K, G\}$ cut-off function".

ii) **(partition of unity)** *Let $\Gamma \subset \mathbb{R}^N$ and $\{G_1, \dots, G_m\}$ be as in Definition 1.2.1. Then*

1) for every G_i there exist $U_i \subset \mathbb{R}^N$ open such that $U_i \Subset G_i$, i.e. "$\overline{U}_i \subset G_i$ and $\overline{U}_i$ is compact", and $\Gamma \subset \bigcup_{i=1}^m U_i$, and

2) there exists a partition of unity subordinate to the covering $\{G_1, \dots, G_m\}$ of Γ.

Proof of i). There exists $\delta > 0$ such that $\{B(x, 3\delta),\ x \in K\} \subset G$. Indeed, if not, there exist a positive sequence of numbers (δ_n) with $\lim_{n\to\infty} \delta_n = 0$, and sequences (x_n) in K and (y_n) in G^c such that $y_n \in B(x_n, 3\delta_n)$. As K is compact, there exists a subsequence of (x_n), denoted again by (x_n), converging to a certain $x_0 \in K$. We still denote by (y_n), resp. (δ_n), the corresponding subsequence of the original sequence (y_n), resp. (δ_n), associated with the subsequence (x_n). Then necessarily $\lim_{n\to\infty} y_n = x_0$ and so $x_0 \in G^c$, because G^c is closed. So $x_0 \in K \cap G^c$, which is a contradiction.

Set $U_\delta = \cup\{B(x, \delta),\ x \in K\}$ and $V_\delta = \cup\{B(x, \delta),\ x \in U_\delta\}$. Clearly $K \subset U_\delta \subset V_\delta \subset G$. For $n \in \mathbb{N}$ define

$$\eta_n(x) = \int_{U_\delta} \rho_n(x - y)dy = \int_{U_\delta \cap B(x, 1/n)} \rho_n(x - y)dy,$$

where (ρ_n) is the mollifier sequence in Example 1.1.13. Clearly $\eta_n \in \mathcal{D}(\mathbb{R}^N)$, $\eta_n \ge 0$ and for n such that $1/n < \delta$ we have

(i) $\eta_n(x) = 1$ for $x \in K$, because $B(x, 1/n) \subset U_\delta$ for all $x \in K$, and

(ii) $\eta_n(x) = 0$ for $x \in V_\delta^c$, because $B(x, 1/n) \cap U_\delta = \emptyset$,

which proves i).

Proof of ii). For $x \in \Gamma$ there exists $i \in \{1, \dots, m\}$ and $\epsilon_x > 0$ such that $B(x, \epsilon_x) \Subset G_i$. As Γ is compact, from its open covering $\{B(x, \epsilon_x),\ x \in \Gamma\}$ there exists a finite covering $\{B(x_j, \epsilon_j),\ j = 1, \dots, M\}$ of Γ with $B(x_j, \epsilon_j) \Subset G_{i_j}$ for a certain $i_j \in \{1, \dots, m\}$. Then $U_i := \bigcup\{B(x_j, \epsilon_j),\ B(x_j, \epsilon_j) \Subset G_i\}$, $i = 1, \dots, m$, satisfies the claim ii.1).

Let $\eta_i \in \mathcal{D}(\mathbb{R}^n)$, $i = 1, \dots, m$, be a $\{\overline{U}_i, G_i\}$ cut-off function, and set $\eta = \sum_{i=1}^m \eta_i$. Clearly $\eta \ge 1$ in $\bigcup_{i=1}^m \overline{U}_i$. Set $U_0 = \{x \in \mathbb{R}^N,\ \eta(x) > 3/4\}$, $G_0 = \{x \in \mathbb{R}^N,\ \eta(x) > 1/2\}$ and let η_0 be a $\{\overline{U}_0, G_0\}$ cut-off function. Then

$$\varphi_0 = 1 - \eta_0, \quad \varphi_i = \begin{cases} \dfrac{\eta_i}{\eta}\eta_0 & \textit{in } G_0, \\ 0 & \textit{in } G_0^c, \end{cases} \quad i = 1, \dots, m,$$

is the desired partition.[6] □

1.2.2 Domains and $C^k(\partial\Omega)$ spaces

To characterize the regularity of an open set Ω, we introduce the following definitions; see, for example, [20].

Notations 1.2.3 *For $x \in \mathbb{R}^N$, we write $x = (x', x_N)$, with $x' = (x_1, \dots, x_{N-1})$, and set*

$$\begin{aligned}
|x| &= (x_1^2 + \cdots + x_{N-1}^2 + x_N^2)^{1/2}, \quad |x'| = (x_1^2 + \cdots + x_{N-1}^2)^{1/2},\\
\mathbb{R}_+^N &= \{x \in \mathbb{R}^N,\ x_N > 0\},\\
Q &= \{x = (x', x_N) \in \mathbb{R}^N,\ |x'| < 1,\ -1 < x_N < 1\},\\
Q^+ &= \{x = (x', x_N) \in \mathbb{R}^N,\ |x'| < 1,\quad 0 < x_N < 1\},\\
Q^0 &= \{x = (x', x_N) \in \mathbb{R}^N,\ |x'| < 1,\qquad x_N = 0\}.
\end{aligned}$$

Definition 1.2.4 (C^kdomains) *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. We say Ω is of class C^k, $k \in \mathbb{R}$, $k \geq 1$, if for every $x \in \partial\Omega$ there exists an open set $G_x \subset \mathbb{R}^N$, $x \in G_x$, and a one-to-one map θ_x satisfying*

i) $\theta_x \in C^k(\overline{Q}; \overline{G}_x)$, $\theta_x^{-1} \in C^k(\overline{G}_x; \overline{Q})$,

ii) $\theta_x(Q^+) = G_x^+ := \Omega \cap G_x$,

iii) $\theta_x(Q^0) = G_x^0 := \partial\Omega \cap G_x$.

This definition states that a C^k domain is more than i) and iii), which one could think at a first attempt. The feature ii) ensures that Ω is on one side of $\partial\Omega$.

Some properties of functions in Sobolev spaces hold for Lipschitz domains which have a weaker regularity than C^k domains.

Definition 1.2.5 (Lipschitz domains) *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. We say Ω is Lipschitz if Ω satisfies Definition 1.2.4 except i), which is replaced by*

$i)_L$ $\quad \exists L_x \geq 0,\ \ \forall x^1, x^2 \in Q \quad L_x^{-1}|x^1 - x^2| \leq |\theta_x(x^1) - \theta_x(x^2)| \leq L_x|x^1 - x^2|$ *(in this case we say "θ_x is a bi-Lipschitz function with Lipschitz bound L_x").*

Remark 1.2.6 *An equivalent formulation of Definition 1.2.4, resp. 1.2.5, is as follows: Ω, an open and bounded set in $\mathbb{R}^N$, is of class C^k, resp. Lipschitz, if for every $x \in \partial\Omega$ there exists an open set G_x containing x such that $\partial\Omega \cap G_x$ is the graph of a C^k, resp. Lipschitz, function of $N-1$ variables (in a local system of coordinates), and $\Omega \cap G_x$ is on one side of $\partial\Omega \cap G_x$.*

Also, more generally, an open set Ω (not necessarily bounded) is said to be minimally smooth, see [48], if there exists G_i, $i \in I \subseteq \mathbb{N}$, open sets, $r_0 > 0$, $m \in \mathbb{N}$ and $L > 0$ such that

[6] Of course, instead of 1/2, 3/4 one could choose any a, b with $0 < a < b < 1$.

i) *if* $x \in \partial\Omega$ *then* $B(x, r_0) \subset G_i$, *for a certain* $i \in I$,

ii) $G_{i_1} \cap \cdots \cap G_{i_{m+1}} = \emptyset$, *for every arbitrary choice of distinct* $i_1, \ldots, i_{m+1} \in I$,

iii) $G_i \cap \Omega = G_i \cap \Omega_i$, *for every* $i \in I$, *where* Ω_i *is a domain that up to a rotation is above the graph of a Lipschitz function* $\psi_i : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ *with Lipschitz bound* L.

Definition 1.2.7 **($C^k(\partial\Omega)$ spaces)** *Assume* Ω *is an open bounded set of class* C^k, $k \geq 1$. *Let* $\{(G_i, \theta_i), i = 1, \ldots, m\}$ *be as in Definition 1.2.4, corresponding to a certain* $\{x_i \in \partial\Omega,\ i = 1, \ldots, m\}$ *such that* $\{G_i, i = 1, \ldots, m\}$ *forms on open covering of* $\partial\Omega$. *Let also* $\{\varphi_0, \varphi_1, \ldots, \varphi_m\}$ *be a partition of unity subordinate to the covering* $\{G_i,\ i = 1, \ldots, m\}$ *of* $\partial\Omega$ *so that for every* $u : \partial\Omega \mapsto \mathbb{R}$ *we have* $u = \sum_{i=1}^{m}(u\varphi_i)$ *in a neighborhood of* $\partial\Omega$. *The space* $C^k(\partial\Omega)$ *is defined by*

$$C^k(\partial\Omega) \;=\; \left\{u : \partial\Omega \mapsto \mathbb{R},\ \ (u\varphi_i) \circ \theta_i(\cdot, 0) \in C_0^k(Q^0),\ \ \forall i = 1, \ldots, m\right\}. \tag{1.2.3}$$

Remark 1.2.8 *The space* $C^k(\partial\Omega)$ *is a Banach space when equipped with the norm*

$$\|u\|_{C^k(\partial\Omega)} = \sum_{i=1,\ldots,m} \|(u\varphi_i) \circ \theta_i(\cdot, 0)\|_{C_0^k(Q^0)}. \tag{1.2.4}$$

It can be shown that two $C^k(\partial\Omega)$ *norms,* $n(\cdot)$, *resp.* $\tilde{n}(\cdot)$, *corresponding to the choice* $\{(G_i, \theta_i, \varphi_i),\ i = 1, \ldots, m\}$, *resp.* $\{(\tilde{G}_i, \tilde{\theta}_i, \tilde{\varphi}_i),\ i = 1, \ldots, \tilde{m}\}$, *are equivalent. Indeed*

$$(u\varphi_i) \circ \theta_i(\cdot, 0) = \sum_{j=1}^{\tilde{m}} ((u\tilde{\varphi}_j) \circ \tilde{\theta}_j(\cdot, 0)) \circ h_{i,j}(\cdot, 0))(\varphi_i \circ \theta_i(\cdot, 0)),$$

where $h_{i,j}(\cdot, 0) = \tilde{\theta}_j^{-1} \circ \theta_i(\cdot, 0)$, *which implies* $n(u) \leq C\tilde{n}(u)$, *with* $C > 0$ *depending only on* $\{(G_i, \theta_i, \varphi_i),\ i = 1, \ldots, m\}$ *and* $\{(\tilde{G}_i, \tilde{\theta}_i, \tilde{\varphi}_i),\ i = 1, \ldots, \tilde{m}\}$. *We proceed similarly to prove the reverse inequality.*

1.3 Review of some important results

The readers are not expected to know the following results, but they must get familiar with them before we consider Sobolev spaces.

1.3.1 Some results from $L^p(\Omega)$ spaces

We will review very briefly a number of results related to $L^p(\Omega)$ spaces. For a detailed analysis of $L^p(\Omega)$ spaces we refer the interested reader to, for example [5, 21, 28, 29, 34].

Definition 1.3.1 ($L^p(\Omega)$ spaces) *Let $p \in [1,\infty]$, $\Omega \subset \mathbb{R}^N$ measurable for the N-dimensional Lebesgue measure dx and define*

$$\begin{cases} L^p(\Omega) = \left\{u : \Omega \mapsto \mathbb{R},\ u\ dx\text{-measurable and } \displaystyle\int_\Omega |u(x)|^p dx < \infty\right\},\ p \in [1,\infty), \\ L^\infty(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ u\ dx\text{-measurable and } \exists M > 0,\ |u(x)| \le M,\ a.a.\ x \in \Omega\}, \\ L^p_{loc}(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ u \in L^p(\omega),\ \forall \omega \subset \Omega,\ \omega\, open,\ \omega \Subset \Omega\}. \end{cases} \tag{1.3.1}$$

The space $L^p(\Omega)$ is equipped with the norm

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\displaystyle\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \inf\{M \in \mathbb{R},\ |u(x)| \le M,\ a.a. x \in \Omega\}, & p = \infty. \end{cases} \tag{1.3.2}$$

Whenever there is no confusion, instead of $L^p(\mathbb{R}^N)$ we will write L^p.

Now we list some important results in L^p spaces.

Theorem 1.3.2 (Fisher-Riesz) *The spaces $L^p(\Omega)$, $p \in [1,\infty]$, are vector spaces. Equipped with the norm $\|\cdot\|_{L^p(\Omega)}$, they are Banach spaces.*

Note that a Banach space is said to be reflexive if $E'' = E$.[7]

Theorem 1.3.3 (Reflexivity of L^p spaces) *For every $p \in (1,\infty)$, the space $L^p(\Omega)$ is reflexive. The spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive. The dual space of $L^1(\Omega)$ is $L^\infty(\Omega)$ and the dual space of $L^\infty(\Omega)$ is a space, called "space of Radon measures" and denoted by $\mathcal{M}(\Omega)$, strictly including $L^1(\Omega)$.*

In the context of reflexivity, we have the following theorem.

Theorem 1.3.4 (Riesz representation theorem in L^p) *Let $p \in [1,\infty)$ and $p' \in (1,\infty]$, the conjugate*[8] *of p, satisfying $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Then we have*

$$\begin{cases} \forall \ell \in (L^p(\Omega))',\ \exists u \in L^{p'}(\Omega) \text{ with } \dfrac{1}{p} + \dfrac{1}{p'} = 1 \text{ such that} \\ \langle \ell, \varphi\rangle_{(L^p(\Omega))' \times L^p(\Omega)} = \displaystyle\int_\Omega u(x)\varphi(x)dx,\ \forall \varphi \in L^p(\Omega), \quad \|\ell\|_{(L^p(\Omega))'} = \|u\|_{L^{p'}(\Omega)}. \end{cases} \tag{1.3.3}$$

The following result shows that L^p functions are not very irregular, in the sense that they can be approximated by $\mathcal{D}$ functions.

[7]In the sense that there exists an isomorphism from E to E''.

[8]In general, the conjugate p' of p is defined for all $p \in [1,\infty]$.

Theorem 1.3.5 (separability and density of $\mathcal{D}$ in L^p) *We have*

i) For every $p \in [1, \infty)$, the space $L^p(\Omega)$ is separable.[9]

ii) The space $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$, $p \in [1, \infty)$. In particular, let (ρ_n) be a mollifier sequence, $u \in L^p(\mathbb{R}^N)$, $p \in [1, \infty)$, and set

$$u_n(x) = (u * \rho_n)(x) := \int_{\mathbb{R}^N} \rho_n(x-y)u(y)dy = \int_{\mathbb{R}^N} \rho_n(y)u(x-y)dx.$$

Then (u_n) is in $L^p(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ and $\lim\limits_{n\to\infty} u_n = u$ in $L^p(\mathbb{R}^N)$.

iii) $L^\infty(\Omega)$ is not separable and $C^0(\Omega)$ is not dense in $L^\infty(\Omega)$.

Theorem 1.3.6 (Lebesgue dominated convergence theorem, Lebesgue DCT) *Let (u_n) be a sequence of functions in $L^1(\Omega)$, $u : \Omega \mapsto \mathbb{R}$ and $v \in L^1(\Omega)$ such that*

i) $\lim\limits_{n\to\infty} u_n(x) = u(x)$ for a.a. $x \in \Omega$,

ii) $|u_n(x)| \leq v(x)$ for all n and a.a. $x \in \Omega$.

Then $u \in L^1(\Omega)$ and $\lim\limits_{n\to\infty} \|u_n - u\|_{L^1(\Omega)} = 0$.

The following is a sort of reciprocal of the theorem above.

Theorem 1.3.7 *Let (u_n) be a sequence in $L^p(\Omega)$, $\lim\limits_{n\to\infty} u_n = u$ in $L^p(\Omega)$, $p \in [1, \infty]$. Then there exists a subsequence (u_{n_k}) of (u_n) and $v \in L^p(\Omega)$ such that*

i) $\lim\limits_{k\to\infty} u_{n_k}(x) = u(x)$ for a.a. $x \in \Omega$,

ii) $|u_{n_k}(x)| \leq v(x)$ for all k and a.a. $x \in \Omega$.

We have also this weak compactness result in $L^p(\Omega)$ spaces; see [5, 29].

Theorem 1.3.8 (Weak compactness in L^p) *Let $\Omega \subset \mathbb{R}^N$ be an open set, $p \in (1, \infty)$ and (u_n) a bounded sequence in L^p, i.e. there exists $C > 0$ such that $\|u_n\|_{L^p(\Omega)} \leq C$ for all n. Then there exist $u \in L^p(\Omega)$ and a subsequence (u_{n_k}) of $U_n)$ converging "weakly to uin $L^p(\Omega)$", i.e.*

$$\lim_{k\to\infty} \int_\Omega u_{n_k}\varphi dx = \int_\Omega u\varphi, \quad \forall \varphi \in L^{p'}(\Omega), \;\; \frac{1}{p} + \frac{1}{p'} = 1.$$

The following result gives an integral criteria for an L^1 function to be zero.

Lemma 1.3.9 (Fundamental lemma) *Let $\Omega \subset \mathbb{R}^N$ be open and $f \in L^1(\Omega)$ such that $\int_\Omega f(x)\varphi(x)dx = 0$ for every $\varphi \in \mathcal{D}(\Omega)$. Then $f = 0$.*

[9] A Banach space E is said to be separable if it contains a dense countable set F, i.e. $F \subset E$ and for every $u \in E$ there exists (φ_n) in F such that $\lim_{n\to\infty} \|\varphi_n - u\|_E = 0$.

1.3.2 Some results from (Functional) Analysis

For the proof of the following results, see, for example, [7, 8, 28, 34].

Theorem 1.3.10 (Contraction mapping theorem) *Let (M, d) be a complete metric space*[10] *and $T : M \mapsto M$ a contraction, i.e. there exists $\kappa \in [0, 1)$ such that*

$$\forall x, y \in M, \quad d(Tx, Ty) \leq \kappa d(x, y).$$

Then T has a unique fixed point in M, i.e. the equation $Tz = z$ has a unique solution $z \in M$. Furthermore, for every $x \in M$, $\lim_{n\to\infty} d(T^n x, z) = 0$.

Theorem 1.3.11 (Inverse mapping theorem) *Let X, Y be Banach spaces, $U \subset X$ open, and $f \in C^k(U; Y)$, $k \in \mathbb{N}$, $k \geq 1$. Furthermore, let $x_0 \in U$ and assume that $f'(x_0) \in \mathcal{L}(X; Y)$, the derivative of f at x_0, is an isomorphism from X to Y, i.e. $f'(x_0) : h \in X \mapsto f'(x_0; h) \in Y$ is continuous and invertible. Then f is a local C^k diffeomorphism at x_0 from X to Y, i.e. there exists $U_0 \subset X$ open with $x_0 \in U_0$, $V_0 \subset Y$ open with $f(x_0) \in V_0$ such that $f^{-1} : V_0 \mapsto U_0$, the inverse of $f : U_0 \mapsto V_0$, exists and $f^{-1} \in C^k(V_0; X)$.*

As a corollary of the inverse mapping theorem we have the implicit mapping theorem, which is important in applications.

Theorem 1.3.12 (Implicit mapping theorem) *Let X, Y, Z be Banach spaces, $U \subset X$, $V \subset Y$ open sets, and $f \in C^k(U \times V; Z)$, $k \in N$, $k \geq 1$. Assume furthermore that $f(x_0, y_0) = 0$ for some $(x_0, y_0) \in U \times V$, and $\partial_y f(x_0, y_0) \in \mathcal{L}(Y; Z)$, the partial derivative of f with respect to y at (x_0, y_0), defines an isomorphism from Y to Z. Then there exists an open set $U_0 \subset U$, $x_0 \in U_0$, and a unique $g \in C^k(U_0; Y)$ such that*

$$y_0 = g(x_0) \ \text{ and } \ f(x, g(x)) = 0, \quad \forall x \in U_0.$$

1.3.3 An application: Ordinary Differential Equations

In this section we review some results about the existence, uniqueness, stability, and regularity of solutions to ODEs. Some of these results will be used in chapter 3 when studying first-order PDEs. For the proof of the following results, see section 9.1.3, or for a detailed analysis see, for example, [28, Chapter XIV, §3].

Let $I \subset \mathbb{R}$ be an interval, $0 \in I$, X be a Banach space[11], $U \subset X$ an open set, and $f \in C^0(U; X)$. Let $x \in U$ and look for $\alpha : I \times U \mapsto U$ such that $\alpha(\cdot, x) \in C^1(I; X)$ and

$$\alpha'(t, x) = f(\alpha(t, x)), \quad t \in I, \tag{1.3.4a}$$

$$\alpha(0, x) = x. \tag{1.3.4b}$$

[10] A metric space (M, d) is said to be complete if every Cauchy sequence in M converges to an element of M.

[11] All along this section the reader may take $X = \mathbb{R}^N$

Here $'$ denotes the derivative with respect to the variable t and x is the initial condition. The function f is sometimes called an "*(autonomous) vector field*". The function α is called an "*integral curve with initial condition* x". Regarding the existence and uniqueness of (1.3.4), we have the following result.

Theorem 1.3.13 *Let $(X, \|\cdot\|_X)$ be a Banach space, $U \subset X$ open, and $f : U \mapsto X$ Lipschitz with a Lipschitz constant L. Let $x_0 \in U$, $\rho \in (0,1)$ such that $\overline{B}(x_0, 2\rho) \subset U$ and $K > 0$ such that $\|f\|_{C^0(\overline{B}(x_0,2\rho))} \leq K$. Finally let $0 < r < \min\{1/L, \rho/K\}$ and set $\overline{I}_r = [-r, r]$. Then there exists $\alpha : \overline{I}_r \times B(x_0, \rho) \mapsto U$, such that for every $x \in B(x_0, \rho)$ there exists a unique solution $\alpha(\cdot, x) \in C^1(\overline{I}_r; X)$ of (1.3.4). Furthermore, if $f \in C^k(U; X)$ then $\alpha(\cdot, x) \in C^{k+1}(\overline{I}_r; X)$.*

Proof. See section Theorem 9.1.4 in Annex. □

The following theorem shows that the solution $\alpha(t, x)$ to (1.3.4) is stable with respect to x, in the sense of (1.3.5). See Theorem 9.1.6 for the proof of this result.

Theorem 1.3.14 *Assume the conditions of Theorem 1.3.13 hold. Let $x \in \overline{B}(x_0, \rho)$ and $\alpha(\cdot, x) \in C^1(\overline{I}_r; U)$ as given by Theorem 1.3.13. Then $x \mapsto \alpha(\cdot, x)$ is uniformly Lipschitz from $\overline{B}(x_0, \rho)$ to $C^0(\overline{I}_r; X)$, i.e. for every x, $y \in \overline{B}(x_0, \rho)$ we have*

$$\|\alpha(\cdot, x) - \alpha(\cdot, y)\|_{C^0(\overline{I}_r;X)} \leq L\|x - y\|_X. \tag{1.3.5}$$

Theorem 1.3.13 shows that if f is of class C^k then $\alpha(\cdot, x)$ is of class C^{k+1} with respect to t. The following theorem shows that $\alpha(\cdot, \cdot)$ is also of class C^k with respect to (t, x). The reader can find the proof, for example, in [28, Chapter XIV, §3].

Theorem 1.3.15 *Assume $f \in C^k(U; X)$, $k \in \mathbb{N} \cup \{\infty\}$. For $x \in U$ let $J(x) \subset \mathbb{R}$ be the interval such that a solution $\alpha(\cdot, x) \in C^{k+1}(J(x); U)$ exists, and let $D = \{(t, x) \in \mathbb{R} \times U,\ \ t \in J(x),\ x \in U\}$. Then $\alpha \in C^k(D; X)$.*

Remark 1.3.16 *Note that often we are interested in a more general problem than (1.3.4), namely*

$$\beta'(\tau, y) = g(\tau, \beta(\tau, y)), \quad \tau \in J, \tag{1.3.6a}$$

$$\beta(s, y) = y, \tag{1.3.6b}$$

where $y \in V$ with $V \subset Y$ open and Y a Banach space, $J \subset \mathbb{R}$ an open interval, $s \in J$, $\beta(\cdot, y) \in C^1(J; Y)$, and $g \in C^0(J \times V; Y)$ (in this case g is called a "non-autonomous vector field"). In fact, (1.3.6) can be written in the form (1.3.4), which is the reason for considering only the problem (1.3.4). Indeed, let $\beta(\cdot, y) \in C^1(J; Y)$. Set $X = \mathbb{R} \times Y$, with $I = \{\tau - s,\ \tau \in J\}$, $U = I \times V$, and consider

$$x \in U,\ \ x = (s, y),$$

$$\begin{aligned} \alpha : I \times U \mapsto U, \quad & \alpha(\cdot, x) \in C^1(I; X), \\ & \alpha(t,x) := (s+t, \beta(s+t,y)) \in U, \\ f \in C^0(U;X), \quad & f(x) := (1, g(x)) = (1, g(s,y)), \quad \textit{so that} \\ & f(\alpha(t,x)) := (1, g(\alpha(t,x)) = (1, g(s+t, \beta(s+t,y))). \end{aligned}$$

Then $\alpha(t,x)$ solves

$$\alpha'(t,x) = (1, \beta'(s+t,y)) = (1, g(s+t, \beta(s+t,y))) = f(\alpha(t,x)), \quad t \in I, \tag{1.3.7a}$$
$$\alpha(0,x) = (s,y) = x, \tag{1.3.7b}$$

if and only if $\beta(\cdot, y)$ solves (1.3.6).

It follows that all the results proved for the solution $\alpha(t,x)$ to (1.3.4) hold for the solution $\beta(\tau, y)$ to (1.3.6), with the assumptions for g in (1.3.6a) such that f in (1.3.7a) satisfies the assumptions of problem (1.3.4). For example, we note that

a) "g is Lipschitz with respect to β with Lipschitz constant L" is equivalent to "f is Lipschitz with constant Lipschitz L", and

b) "$\|f\|_{C^0(\overline{B}(x_0,2\rho)} \leq 1+K$" is equivalent to "$\|g\|_{C^0(\overline{B}(s_0,y_0),2\rho)} \leq K$" where $x_0 = (s_0, y_0)$, $s_0 \in J$, $y_0 \in V$, provided $\overline{B}(x_0, 2\rho) = \overline{B}((s_0,y_0),2\rho) \subset J \times V$.

Then (1.3.6) has a unique $C^1(\overline{J}_r; Y)$ solution $\beta = \beta(\cdot, y)$, with $\overline{J}_r = [s_0 - r, s_0 + r]$, $0 < r < \min\left\{\frac{1}{L}, \frac{\rho}{1+K}\right\}$ for every $(s,y) \in \overline{B}((s_0,y_0),\rho)$. □

We are interested in the following questions for the problem (1.3.4):

$$\begin{cases} i) & \textit{Does it have a solution?} & \textit{[existence]} \\ ii) & \textit{Is the solution unique?} & \textit{[uniqueness]} \\ iii) & \textit{Is the solution stable, say like in } (1.3.5)\textit{?} & \textit{[stability].} \end{cases} \tag{1.3.8}$$

Similarly for PDEs, we are interested in questions i), ii), and iii). Given that usually the solutions to PDEs are "*weak*", meaning that they do not have the classical regularity C^k (we will see what it means precisely in the following chapters), for PDEs we consider also the following question:

$$iv) \quad \textit{How regular is the solution?} \qquad \textit{[regularity]} \tag{1.3.9}$$

We will address i) and ii) for all the PDEs we consider in this book. For ease of reading, in section 9.6 in Annex we have addressed question iv) for second-order linear elliptic PDEs.

Problems

Problem 1.1 *Let X, Y, and Z be three Banach spaces, $f \in C^0(X;Y)$, $g \in C^0(Y;Z)$ with $Im(f) \subset Dom(g)$, where, in general, for a function $f : X \mapsto Y$ we set*

$$dom(f) = \{x \in X, \ \ f(x) \ \ is \ well\text{-}defined\}, \quad Im(f) = \{f(x), \ x \in dom(f)\}.$$

i) Prove that $g \circ f \in C^0(X;Z)$, where $g \circ f : X \mapsto Z$, $(g \circ f)(x) = g(f(x))$.

ii) Assume $f \in C^1(X;Y)$, $g \in C^1(Y;Z)$. Prove[12] that $g \circ f \in C^1(X;Z)$ and for $h \in X$ we have $(g \circ f)'(x;h) = g'(f(x);f'(x;h))$.

Problem 1.2 *Find the norm of the elements of $\mathcal{L}(\mathbb{R}^{nN})$; see Example 1.1.5, Remark 1.1.4.*

Problem 1.3 *Let $\Omega \subset \mathbb{R}^N$ be open, $k \in \mathbb{N}$, and $\lambda \in (0,1]$. Prove that $\|\cdot\|_{C_b^k(\Omega)}$, resp. $\|\cdot\|_{C_b^{k,\lambda}(\Omega)}$, as given by (1.1.20), resp. (1.1.21), defines a norm in $C_b^k(\Omega)$, resp. $C_b^{k,\lambda}(\Omega)$.*

Problem 1.4 *Find a C^∞ domain in $\mathbb{R}^N$, and another one which is C^1 but not C^2.*

Problem 1.5 *i) Show that if $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain then it is minimally smooth. ii) Find an unbounded domain $\Omega \subset \mathbb{R}^N$ which is minimally smooth.*

Problem 1.6 *Prove the equivalence in Remark 1.2.6.*

Problem 1.7 *Show that if $p \in (0,1)$ then $\|\cdot\|_{L^p(\Omega)}$ is not a norm by showing that the triangle inequality is not satisfied.*

Problem 1.8 *Show that Theorem 1.3.6 does not hold if only one of i), ii) is satisfied.*

[12] One may use this fact: f differentiable at x is equivalent to $f(x+h) = f(x) + f'(x;h) + \epsilon_h \|h\|_X$ for all $x \in X$, where $\epsilon_h \in Y$ is such that $\lim_{\|h\|_X \to 0} \|\epsilon_h\|_Y = 0$.

2. Partial differential equations

Infinitesimal calculus is one of the most important mathematical developments of the seventeenth century. The need for better understanding and prediction of the behavior of real-world phenomena, such as the motion of planets or the trajectory of a particle, for example, motivated mathematicians to consider (systems of) Ordinary Differential Equations (ODEs). The practical interest on more complex phenomena, such as the distribution of heat, of an electric or magnetic field, or the motion of fluids, made the understanding of PDEs and their rigorous analysis a central point of mathematical analysis and the birthplace of the development of new areas of mathematics, such as Calculus of Variations and Functional Analysis.

Very fundamental results about linear PDEs were achieved around the end of the nineteenth century and during the twentieth century, for example results about the existence and uniqueness of weak and strong solutions, and the regularity of weak solutions to linear and certain nonlinear PDEs. Due to the explosion of the number of applications, the interest in PDEs continues to grow and is more oriented toward nonlinear PDEs.

It would be useful to classify PDEs into classes such that the PDEs of the same class share the same qualitative properties. For example, a classification such that the proof of the existence, uniqueness, regularity, and even numerical analysis is the same (or similar) for all the PDEs of the same class. If restricted to linear second-order PDEs, a complete classification can be done by using Fourier transform. We will not enter into details and suggest the interested reader [12]; see also Definition 2.0.3 for the case of a second-order linear PDE with constant coefficients.

Here we will simply define linear and nonlinear PDEs. This definition serves as a basis for identifying types of PDEs, but in general the PDEs of the same type under this definition do not share the same qualitative properties.

Definition 2.0.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set.*

i) A (N dimensional) PDE of order $m \in \mathbb{N}$ in Ω is an equation of an unknown function $u = u(x)$, of its derivatives up to order m and of x of the form

$$E(x, u, D^1 u, \dots D^m u) = 0, \quad x \in \Omega, \tag{2.0.1}$$

where $E : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \times \cdots \times \mathbb{R}^{N^m} \mapsto \mathbb{R}$ and $D^k u$ given by (1.1.14).

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6_2

ii) Usually a PDE is equipped with boundary conditions of the form

$$B(x, u, D^1u, \dots, D^{m_b}u) = 0,\ x \in \partial\Omega, \tag{2.0.2}$$

where $m_b \le m-1$ *and* $B : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \times \cdots \times \mathbb{R}^{N^{m_b}} \mapsto \mathbb{R}$.

iii) A function u is called a "classical solution to the problem (2.0.1), (2.0.2)" if $u \in C^m(\Omega) \cap C^{m_b}(\overline{\Omega})$ *and satisfies (2.0.1), (2.0.2) pointwise.*

iv) If (2.0.1), resp. (2.0.2), is linear with respect to u and all its derivatives then (2.0.1), resp. (2.0.2), is called "linear". Otherwise it is called "nonlinear".

Example 2.0.2 *Let* $A = (a_{i,j}) \in \mathbb{R}^{N\cdot N}$ *be a symmetric matrix,* $b = (b_i) \in \mathbb{R}^N$, $c \in \mathbb{R}$, $u = u(x)$ *a smooth real function,* $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ *and consider the PDE*

$$Lu := -\sum_{i,j=1}^{N} a_{i,j}\partial^2_{x_i x_j}u + \sum_{i=1}^{n} b_i \partial_{x_i}u + cu = f, \quad \textit{with } f \textit{ given.} \tag{2.0.3}$$

Let $A = V \cdot \Lambda \cdot {}^tV$ *be the spectral decomposition of* A, *with* $\Lambda = \operatorname{diag}(\lambda_i)$ *the diagonal matrix of eigenvalues of* A, $V = (v_{i,j})$ *the matrix having for columns the eigenvectors of* A *with*[1] ${}^tV = V^{-1}$. *Let* $x = V \cdot \xi$, $\xi = {}^tV \cdot x$ *and define* $v(\xi) := u(x)$. *Then*

$$\begin{aligned}
\partial_{x_i}u &= \sum_{k=1}^{N} \partial_{\xi_k}v\partial_{x_i}\xi_k = \sum_{k=1}^{N} v_{i,k}\partial_{\xi_k}v, \qquad \textit{(we used } \partial_{x_i}\xi_k = v_{i,k}\textit{)} \\
\partial^2_{x_i x_j}u &= \sum_{k,l=1}^{N} v_{i,k}v_{j,l}\partial_{\xi_k\xi_l}v; \quad \textit{so} \\
\sum_{i,j=1}^{N} a_{i,j}\partial^2_{x_i x_j}u &= \sum_{i,j=1}^{N} a_{i,j}\sum_{k,l=1}^{N} v_{i,k}v_{j,l}\partial_{\xi_k\xi_l}v = \sum_{k,l=1}^{N}\left(\sum_{i,j=1}^{N}(v_{i,k}\cdot a_{i,j}\cdot v_{j,l})\right)\partial^2_{\xi_k\xi_l}v \\
&= \sum_{k,l=1}^{N} ({}^tV \cdot A \cdot V)_{k,l}\partial^2_{\xi_k\xi_l}v = \sum_{k,l=1}^{N} \Lambda_{k,l}\partial^2_{\xi_k\xi_l}v \\
&= \sum_{k=1}^{N} \lambda_k \partial^2_{\xi_k\xi_k}v, \\
b \cdot \nabla u &= \sum_{k=1}^{N} \mu_k \partial_{x_k}v, \quad \mu_k = \sum_{i=1}^{N} v_{i,k}b_i; \quad \textit{hence} \\
Lu &= -\sum_{k=1}^{N} \lambda_k \partial^2_{\xi_k\xi_k}v + \sum_{k=1}^{N} \mu_k\partial_{\xi_k}v + cv.
\end{aligned}$$

[1] For a matrix $M \in \mathbb{R}^{N\times N}$, ${}^tM$ will denote its transposed matrix and M^{-1} will denote its inverse matrix.

Note that if $\eta = \operatorname{diag}(|\lambda_k|^{-1/2}) \cdot \xi$ *and* $w(\eta) = v(\xi)$*, we get*

$$Lu = -\sum_{k=1}^{N} \alpha_k \partial^2_{\eta_k \eta_k} w + \sum_{k=1}^{N} \beta_k \partial_{\eta_k} w + cw, \tag{2.0.4}$$

with $\alpha_k \in \{\pm 1, 0\}$*,* $\beta_k = \mu_k |\lambda_k|^{-1/2}$.

Based on the form (2.0.4), the following definition gives a detailed classification of second-order linear PDEs with constant coefficients.

Definition 2.0.3 *Consider the PDE* (2.0.3) *and let* λ_i*,* $i = 1, \ldots, N$*, be the eigenvalues of* A*. We say the PDE* (2.0.3) *is*

"elliptic",	*if all* λ_i *have the same sign,*
"hyperbolic",	*if all* λ_i *have the same sign, except one which has the opposite sign,*
"parabolic",	*if all* λ_i *have the same sign except one which is zero.*

It turns out that when (2.0.3) is elliptic (resp. hyperbolic, parabolic) then in (2.0.4) all α_i have the same sign (resp. all α_i have the same sign except one which has the opposite sign, all α_i have the same sign except one which is zero).

Example 2.0.4 *Let* $u = u(x,t)$*,* $x, t \in \mathbb{R}$ *and consider the second-order PDE*

$$au_{tt} + 2bu_{tx} + cu_{xx} = 0, \tag{2.0.5}$$

with $a, b, c \in \mathbb{R}$*,* $b^2 - ac > 0$*. Assume*[2] $a^2 + c^2 \neq 0$*, for example* $c \neq 0$*. Then* (2.0.5) *is of the form* (2.0.3) *with* $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$*, whose product of eigenvalues is* $ac - b^2 < 0$*, so* (2.0.5) *is hyperbolic. Furthermore, the solution is given by*

$$u(x,t) \quad = \quad A\left(x + \frac{ct - bx}{\sqrt{b^2 - ac}}\right) + B\left(x - \frac{ct - bx}{\sqrt{b^2 - ac}}\right), \tag{2.0.6}$$

with A *and* B *arbitrary* C^2 *functions. In particular, if* $a = 1$*,* $b = 0$*,* $c = -\omega^2$*, so the original equation is* $u_{tt} - \omega^2 u_{xx} = 0$*, then every solution is of the form*[3]

$$u(x,t) = A(x - \omega t) + B(x + \omega t). \tag{2.0.7}$$

Indeed, proceeding as in Example 2.0.2 it is easy to find the eigenvectors of A*, which leads this change of variables and function* v*:*

$$v(\xi, \tau) = u(x,t), \quad \left\{ \begin{array}{rcl} x &=& \xi, \\ t &=& \xi \dfrac{b}{c} + \tau \dfrac{\sqrt{b^2 - ac}}{c}, \end{array} \right. \quad \text{so} \quad \left\{ \begin{array}{rcl} \xi &=& x, \\ \tau &=& \dfrac{ct - bx}{\sqrt{b^2 - ac}}. \end{array} \right.$$

[2] If $a^2 + c^2 = 0$ then the equation is reduced to $u_{xt} = 0$, so its solution is $u(x,t) = A(x) + B(t)$, with A and B arbitrary C^2 functions.

[3] Or, equivalently $u(x,t) = A(\omega t - x) + B(\omega t + x)$.

It follows by straightforward computations that $v_{\tau\tau} - v_{\xi\xi} = 0$. Next, introducing the new variables $\eta = \xi + \tau$, $s = \xi - \tau$ and the new function $w(\eta, s) = v(\xi, \tau)$ implies that w satisfies $w_{s\eta} = 0$. Therefore $w(\eta, s) = A(\eta) + B(s)$, with A, B arbitrary C^2 functions, $v(\xi, \tau) = A(\xi + \tau) + B(\xi - \tau)$ and therefore (2.0.6) holds.

2.1 Some prototypes of PDEs

In this section we will present some PDEs that have important applications. Although some of them may look similar at first, they have very different properties. Each of them represents the simplest element of a distinct class of PDEs, and typically, the methods we develop for each of them are also used for the PDEs of the same class.

In all the following examples $\Omega \subset \mathbb{R}^N$ is open, $0 < T \leq \infty$, f is a given function, and u is the unknown function. Also, here $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $(x, t) = (x_1, \ldots, x_N, t)$, and the unknown function $u = u(x, t)$ or $u = u(x)$, depending on whether or not the problem involves the variable t.

Transport equation

The transport equation, which sometimes is referred to as the *continuity equation*, models the transport of a certain quantity, for example, the transport of a cloud of particles. In its simplest form it is

$$u_t + b \cdot \nabla u := u_t + \sum_{i=1}^{N} b_i \partial_{x_i} u = f \quad \text{in } \Omega \times (0, T), \quad b = (b_1, \ldots, b_N) \in \mathbb{R}^N. \tag{2.1.1}$$

In chapter 3, we will consider general first-order PDEs, which include the equation 2.1.1.

Laplace equation

The Laplace equation models, for example, the distribution of heat in a stationary regime, i.e. the heat does not depend on time. This is one of the most studied PDEs and the most representative of second-order elliptic PDEs. It has the form

$$-\Delta u := -\left(\partial^2_{x_1 x_1} u + \cdots + \partial^2_{x_N x_N} u\right) = f \quad \text{in } \Omega. \tag{2.1.2}$$

We will study the classical and weak solutions to (2.1.2) in chapters 4 and 7.

Heat equation

The heat equation models the distribution of heat in a time-dependent regime, i.e. the heat depends on time. In its simplest form it is

$$u_t - c^2 \Delta u = f \quad \text{in } \Omega \times (0, T), \quad c > 0. \tag{2.1.3}$$

Equation (2.1.3) is the most representative of second-order PDEs of parabolic type. We will study (2.1.3) in chapter 8.

Wave equation

The wave equation models the oscillations of a certain quantity, for example the motion of an elastic membrane. In it simplest form it has the form

$$u_{tt} - c^2\Delta u = f \quad \text{in } \Omega \times (0,T), \quad c > 0. \tag{2.1.4}$$

Equation (2.1.4) is the most significant representative of second-order PDEs of hyperbolic type. We will study (2.1.4) in chapter 8.

Nonlinear PDEs

There are many nonlinear PDEs, often originating from applications. For example, minimal surface equation in $\mathbb{R}^3$, see [20],

$$(1+u_x^2)u_{yy} - 2u_xu_yu_{xy} + (1+u_y^2)u_{xx} = 0 \;\; in \;\, \Omega, \quad u = g \;\text{ on } \partial\Omega, \tag{2.1.5}$$

where $u = u(x)$, $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$, describes the surface with the least area among all graphs of functions in Ω equaling g on $\partial\Omega$.

Another important example of nonlinear PDEs are incompressible Navier-Stokes equations, see, for example, [52], given by

$$\rho(\partial_t u + (u\cdot\nabla)u) - \mu\Delta u + \nabla p = f \quad in \;\, \Omega \times (0,T), \tag{2.1.6a}$$

$$\nabla\cdot u = 0 \quad in \;\, \Omega \times (0,T), \tag{2.1.6b}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, $0 < T \leq \infty$, $\mu, \rho > 0$ are given physical parameters, $f = (f_1, \ldots, f_N)$ is a given function, and $u = u(x,t)$, $u = (u_1, \ldots, u_N)$ and $p = p(x,t) \in \mathbb{R}$ are unknown. The variable u represents the velocity and p represents the pressure. Here the nonlinearity is due to the term $(u\cdot\nabla)u$.

The analysis of nonlinear PDEs is more difficult than linear PDEs. However, typically the question of the existence of solutions relies strongly on the existence of solutions to certain associated linear PDEs, on compactness results, and on tools of Functional Analysis such as fixed point or topological degree theorems; see, for example, [8, 20, 42]. We will discuss very briefly some nonlinear PDEs mostly in the context of examples; see Section 7.3.

Problems

Problem 2.1 *Find a C^1 solution $u = u(x,y)$ of*

$$\begin{array}{llll} a) & u_x = 3x^2y + y, & u_y = x^3 + x, & u(0,0) = 0, \\ b) & u_x = y\cos x + 1, & u_y = \sin x, & u(0,0) = 0. \end{array}$$

Problem 2.2 *Find a* C^2 *solution* $u = u(x,t)$, $(x,t) \in \mathbb{R} \times (0,\infty)$, *of*

$$\begin{array}{llcl} a) & u_{tt} - u_{xx} & = & \sin t + x^{100}, \\ b) & u_{tt} - u_{xx} & = & tx, \\ c) & u_{tt} + u_{xx} & = & \cos t - x^{100}. \end{array}$$

Show that in fact each of these problems has an infinite number of solutions.

Problem 2.3 *Find all classical solutions to*

$$\begin{array}{llcllll} a) & u_{ttx} - u_{xxx} & = & 0 & \text{in } \mathbb{R} \times (0,\infty), & u_x(x,0) = 0, & u_{tx}(x,0) = \sin x, \quad x \in \mathbb{R}, \\ b) & u_{ttt} - u_{xxt} & = & 0 & \text{in } \mathbb{R} \times (0,\infty), & u_t(x,0) = 0, & u_{tt}(x,0) = \cos x, \quad x \in \mathbb{R}. \end{array}$$

Problem 2.4 *Let* $u = u(x,t)$, $(x,t) \in \mathbb{R} \times (0,\infty)$ *and consider the PDE*

$$u_{tt} + 4u_{tx} + 4u_{xx} + (u_x - 2u_t) = 0.$$

Use the transformation in Example 2.0.2 to show that this equation is of parabolic type, and in the new coordinates (ξ,τ) *the PDE is written in the form* $v_\tau - 5v_{\xi\xi} = 0$.

3. First-order PDEs: classical and weak solutions

First-order PDEs describe many real-world phenomena. Conservation laws, which state that a certain property of a physical system is conserved, are first-order PDEs. Hamilton-Jacobi equations, which describe the motion of a mechanical system as well as the motion of a cloud of particles, are also first-order PDEs.

In this chapter, we will briefly discuss first-order PDEs with the intention of highlighting mainly the method of characteristics and the concepts of classical and weak solutions. For a deeper analysis of the subject, see the pioneering works [6, 9, 30, 31], or [16] for a modern overview.

Let $E : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$, $E = E(x, z, p)$, $p = (p_1, \ldots, p_N)$, $z \in \mathbb{R}$, $x = (x_1, \ldots, x_N)$, $u : \Omega \subset \mathbb{R}^N \mapsto \mathbb{R}$, with Ω open, and consider the problem

$$E(x, u(x), \nabla u(x)) = 0, \quad x \in \Omega, \tag{3.0.1a}$$

$$u(x) = g(x), \quad x \in \Gamma \subset \partial\Omega, \tag{3.0.1b}$$

where Γ is relatively open[1] in $\partial\Omega$. Hereafter, we assume that E, g, *and* Ω *are sufficiently smooth*, unless otherwise specified.

First, we will look for a classical solution to (3.0.1), and next we will demonstrate the idea of weak solutions restricted to scalar conservation laws. In both cases, we will use the method of characteristics, which consists of writing (3.0.1) in the form of a system of ODEs, called the "characteristic equations". These characteristic equations describe the solution u along some curves, called "*characteristic curves*".

Motivated by physical phenomena, such as the motion of a cloud of particles, the idea for solving (3.0.1) is as follows. For a given $x \in \Omega$, there is a particle moving along a trajectory, or a characteristic curve, passing through x; see Fig. 3.0.1. This trajectory eventually connects x with a certain point $y^0 \in \partial\Omega$. If $y^0 \in \Gamma$ then $u(y^0) = g(y^0)$ is known and therefore the characteristic equations associated with y^0 will provide the value of $u(x)$ along the characteristic curve.

[1] So $\Gamma = G \cap \partial\Omega$, where G is a certain open set in $\mathbb{R}^N$.

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6_3

From the description of the method, we have to deal with the following two problems.

i) *Characterize the characteristic curves and the system of characteristic equations associated with* (3.0.1).

ii) *Knowing the solution to characteristic equations, find a solution to the problem* (3.0.1).

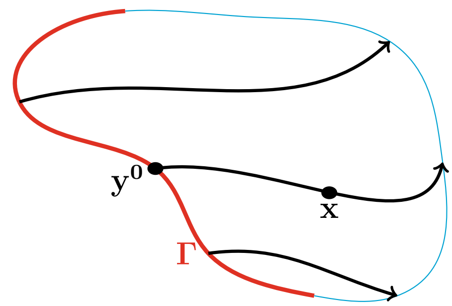

Figure 3.0.1: Characteristic curves

As a first example of a first-order PDE that addresses these two questions, let us consider the transport equation (2.1.1)

$$u_t + b \cdot \nabla u = f \;\; in \;\; \mathbb{R}^N \times (0, \infty), \tag{3.0.2a}$$
$$u = g \;\; on \;\; \mathbb{R}^N \times \{0\}. \tag{3.0.2b}$$

We assume that (3.0.2) has one classical solution $u \in C^0(\mathbb{R}^N \times [0,\infty)) \cap C^1(\mathbb{R}^N \times (0,\infty))$. For a fixed $(x,t) \in \mathbb{R}^N \times (0,\infty)$, we set $z(s) = u(x+sb, t+s)$ and then we obtain

$$\begin{aligned} z'(s) &= u_t(x+sb,t+s) + b \cdot \nabla u(x+sb,t+s) = f(x+sb,t+s), \quad s \in (-t,\infty), \\ z(-t) &= u(x-tb,0) = g(x-tb). \end{aligned}$$

So, z satisfies a differential equation. It implies

$$\begin{aligned} u(x,t) - g(x-tb) &= z(0) - z(-t) = \int_{-t}^{0} z'(s)ds = \int_{-t}^{0} f(x+sb,t+s)ds \\ &= \int_0^t f(x+(s-t)b,s)ds. \end{aligned}$$

Therefore, if $g \in C^1(\mathbb{R}^N)$ and $f \in C^0(\mathbb{R}^N \times [0,\infty))$ then the classical solution u is given by

$$u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b,s)ds. \tag{3.0.3}$$

The following is also true: if $g \in C^1(\mathbb{R}^N)$ and $f, \partial_{x_i} f \in C^0(\mathbb{R}^N \times [0,\infty))$ for all i, then u given by (3.0.3) is a classical solution to (3.0.2).

Note that we have solved (3.0.2) by converting it to an ordinary differential equation of the form $z'(s) = f(x+sb,t+s)$. Such an equation describes the solution u along the curve $\{(x+sb,t+s),\; s \in (-t,\infty)\}$, which is called a "*characteristic curve*". This method is a particular case of the so-called "*method of characteristics*"; see Section 3.1 for more details.

In general, $g \notin C^1(\mathbb{R}^N)$ and $\partial_{x_i} f \notin C^1(\mathbb{R}^N \times [0,\infty))$, and so (3.0.3) does not provide a C^1 "*classical*" solution to (3.0.2). However, u given by (3.0.3) is the only reasonable candidate for a solution to (3.0.2). To give a meaning to the solution u in this case, we introduce the so-called "*weak solution to* (3.0.2)" satisfying

$$\int_{\mathbb{R}^N} g(x)\varphi(x,0)dx + \int_0^\infty\int_{\mathbb{R}^N} (u\varphi_t + u(b\cdot\nabla_x\varphi))dxdt + \int_0^\infty\int_{\mathbb{R}^N} f(x,t)\varphi(x,t)dxdt = 0, \quad (3.0.4)$$

for all $\varphi \in \mathscr{D}(\mathbb{R}^N \times \mathbb{R})$, where $b \cdot \nabla_x \varphi = \sum_{i=1}^N b_i \partial_{x_i}\varphi$. Note that (3.0.4) is obtained by multiplying (3.0.2a) by φ, integrating by parts, and then using (3.0.2b). It is well-defined for all $f \in L^1_{loc}(\mathbb{R}^N \times (0,\infty))$ and $g \in L^1_{loc}(\mathbb{R}^N)$.

3.1 Method of characteristics

In this section and the next one, we will present the method of characteristics and show that under some assumptions, (3.0.1) is equivalent to a system of differential (characteristic) equations, which provides a local classical solution. The results are classical and we follow the pioneering works [6, 9, 19], and [16] for a modern presentation of the results.

Let $\{y = y(t) = (y_1(t), \ldots, y_N(t)),\ t \in I \subset \mathbb{R}\} \subset \Omega$, with I an open interval, be a given smooth curve, $u \in C^1(\Omega)$ and define

$$z(t) := u(y(t)), \quad p(t) = (p_1(t), \ldots, p_N(t)) := \nabla u(y(t)), \quad t \in I. \quad (3.1.1)$$

We have the following result.

Proposition 3.1.1 *Let E be a C^1 function, $u \in C^2(\Omega)$ be a solution to* (3.0.1a), *and z, p defined by* (3.1.1). *If y solves* (3.1.2a) *then $z = z(t)$ and $p = p(t)$ solve* (3.1.2b), (3.1.2c), *where*

$$y' = \nabla_p E(y,z,p), \quad (3.1.2a)$$
$$z' = p \cdot \nabla_p E(y,z,p), \quad (3.1.2b)$$
$$p' = -\nabla_x E(y,z,p) - p\partial_z E(y,z,p). \quad (3.1.2c)$$

Proof. We can differentiate $z(t)$ in (3.1.1), which in combination with p in (3.1.1) and (3.1.2a) gives

$$z'(t) = \sum_{j=1}^N \partial_{x_j} u(y(t)) y_j'(t) = \sum_{j=1}^N p_j(t)\partial_{p_j} E(y(t), z(t), p(t)). \quad (3.1.3)$$

Now, let us differentiate $p(t)$ in (3.1.1), which after using (3.1.2a) gives

$$p_i'(t) \;=\; \sum_{j=1}^N \partial_{x_i x_j} u(y(t)) y_j'(t) = \sum_{j=1}^N \partial_{x_i x_j} u(y(t)) \partial_{p_j} E(y(t), u(y(t)), \nabla u(y(t)))$$

$$= \sum_{j=1}^{N} \partial_{p_j} E(y,u,\nabla u)\partial_{x_i x_j} u, \quad on\ \{y(t),\ t \in I\}. \tag{3.1.4}$$

In order to eliminate second-order derivative terms in (3.1.4), we differentiate (3.0.1a) w.r.t. x_i, $i = 1, \dots, N$, and get

$$\partial_{x_i} E(x,u,\nabla u) + \partial_z E(x,u,\nabla u)\partial_{x_i} u + \sum_{j=1}^{N} \partial_{p_j} E(x,u,\nabla u)\partial_{x_i x_j} u = 0, \quad \forall x \in \Omega. \tag{3.1.5}$$

Then (3.1.4) combined with (3.1.5) evaluated at $x = y(t)$ (so, $u(y(t)) = z(t)$ and $\nabla u(y(t)) = p(t)$) gives

$$p_i'(t) = -\partial_{x_i} E(y(t), z(t), p(t)) - p_i(t)\partial_z E(y(t), z(t), p(t)), \quad t \in I, \tag{3.1.6}$$

which completes the proof. □

Equations (3.1.2) are called "*characteristic equations (associated with* (3.0.1a)", and the curve $\{y(t),\ t \in I\}$ is called a "*characteristic curve*". Proposition 3.1.1 gives a natural method for solving (3.0.1). Namely, it suggests that if (y, z, p) solves (3.1.2) then $u(x) := z(y(t))$, with $x = y(t)$, is a good candidate for the solution to (3.0.1a).

Before we move to the analysis of the method to characteristics, let us see how it works through a number of typical examples. Consider first an example of first-order linear PDEs, which have the general form

$$E(x,u,\nabla u) = a(x) \cdot \nabla u(x) + b(x)u + c(x) = 0, \quad x \in \Omega, \tag{3.1.7}$$

with a, b, and c being smooth functions. So, here $E(x,z,p) = a(x) \cdot p + b(x)z + c(x)$ and then (3.1.2) is equivalent to

$$y' = a(y), \tag{3.1.8a}$$
$$z' = p \cdot a(y) = -(c(y) + b(y)z), \tag{3.1.8b}$$
$$p' = -\nabla_x E(y,z,p) - pb(y), \tag{3.1.8c}$$

because on the characteristic curve we have $E(y,z,p) = 0$, so $p \cdot a(y) = -(c(y) + b(y)z)$. Here we do not need to solve for p as the characteristic equations are closed for y and z.

Example 3.1.2 *Consider*

$$\begin{cases} 2x_1x_2\partial_{x_1} u + \partial_{x_2} u = u & in \quad \Omega = \{(x_1,x_2),\ x_2 > 0\}, \\ u(x_1, 0) = g(x_1) & on \quad \Gamma = \{(x_1,x_2),\ x_2 = 0\}. \end{cases} \tag{3.1.9}$$

This equation is of the form (3.1.7) *with* $a = (2x_1x_2, 1)$, $b = -1$, $c = 0$, *and so* $E(x,z,p) = 2x_1x_2p_1 + p_2 - z$. *According to* (3.1.8), y *and* z *solve*

$$y_1' = 2y_1y_2, \quad y_2' = 1, \quad z' = z, \quad with\ y(0) = (y_1^0, 0) \in \Gamma,\ z(0) = g(y_1^0).$$

The initial conditions for y and z are chosen such that the characteristic curves start on Γ, so $y(0) \in \Gamma$, and that $z(0) = u(y(0))$. We solve this system of ODEs and get

$$y_1(t) = y_1^0 e^{t^2}, \quad y_2(t) = t, \quad z(t) = g(y_1^0)e^t.$$

We note that y and z depend on t and y_1^0, so $y = y(t, y_1^0)$, $z = z(t, y_1^0)$.

Now we "construct" the solution u at $x = (x_1, x_2) \in \Omega$ by assigning to it the value of $z(t, y_1^0)$ with $y(t, y_1^0) = x$. Solving the last equation gives $y_1^0 = x_1 e^{-x_2^2}$, $t = x_2$. Hence

$$u(x_1, x_2) = u(y(t)) = z(t) = g(y_1^0)e^t = g(x_1 e^{-x_2^2})e^{x_2}.$$

It is easy to verify that u is a classical solution to (3.1.9) if g is C^1.

Next we consider an example of so-called first-order quasi-linear PDEs, which have the form

$$E(x, u, \nabla u) = a(x, u(x)) \cdot \nabla u(x) + b(x, u(x)) = 0, \quad x \in \Omega. \tag{3.1.10}$$

So, here $E(x, z, p) = a(x, z) \cdot p + b(x, z)$. Therefore, (3.1.2) is equivalent to

$$y' = a(y, z), \tag{3.1.11a}$$
$$z' = p \cdot a(y, z) = -b(y, z), \tag{3.1.11b}$$
$$p' = -\nabla_x E(y, z, p) - p\partial_z E(y, z, p), \tag{3.1.11c}$$

because on the characteristic curve we have $E(y, z, p) = 0$, so $p \cdot a(y, z) = -b(y, z)$.

Example 3.1.3 *Consider*

$$\left\{ \begin{array}{rcll} u\partial_{x_1} u + \partial_{x_2} u & = & 1, & \text{in} \quad \Omega = \{(x_1, x_2),\ x_1 < x_2 < 0\}, \\ u(x_1, x_1) & = & g(x_1) = \frac{1}{2}x_1 & \text{on} \quad \Gamma = \{(x_1, x_1),\ x_1 < 0\}. \end{array} \right. \tag{3.1.12}$$

This PDE system is of the form (3.1.10) with $a = (z, 1)$, $b = -1$, and so $E(x, z, p) = zp_1 + p_2 - 1$. Applying (3.1.11) and noting that $\nabla_x E = 0$ and $\partial_z E = p_1$, we obtain

$$\left\{ \begin{array}{rcl} y_1' & = & z, \\ y_2' & = & 1, \end{array} \right. \qquad z' = 1, \qquad \left\{ \begin{array}{rcl} p_1' & = & -p_1^2, \\ p_2' & = & -p_1 p_2. \end{array} \right.$$

We note that the system is closed in (y, z), so we do not need to solve for p. The initial conditions for y and z are

$$\left\{ \begin{array}{rcl} y_1(0) & = & y_1^0, \\ y_2(0) & = & y_1^0, \end{array} \right. \qquad z(0) = \frac{1}{2} y_1^0.$$

Then we get

$$\left\{ \begin{array}{rcl} y_1(t) & = & \frac{1}{2}t^2 + \frac{1}{2}y_1^0 t + y_1^0, \\ y_2(t) & = & t + y_1^0, \end{array} \right. \qquad z(t) = t + \frac{1}{2}y_1^0, \quad y_1^0 \in \mathbb{R}.$$

To find $u(x)$ we first find the characteristic curve $y(t, y_0^1)$ passing through $x \in \Omega$, so $y(t, y^0) = x$ or equivalently

$$\left\{\begin{array}{rcl} \frac{1}{2}t^2 + \frac{1}{2}y_1^0 t + y_1^0 &=& x_1, \\ t + y_1^0 &=& x_2, \end{array}\right. \quad \textit{which implies} \quad \left\{\begin{array}{rcl} t &=& 2\dfrac{x_2 - x_1}{2 - x_2}, \\ y_1^0 &=& \dfrac{2x_1 - x_2^2}{2 - x_2}. \end{array}\right.$$

This implies that

$$u(x_1, x_2) = z(t, y_1^0) = t + \frac{1}{2}y_1^0 = \frac{1}{2}\frac{4x_2 - x_2^2 - 2x_1}{2 - x_2}.$$

It is easy to check that u is a classical solution to (3.1.12).

Now we consider an example of a fully nonlinear first-order PDE, which are all first-order PDEs that are neither linear nor quasi-linear. In this case we have to integrate the full system of differential equations (3.1.2), which in general is not possible.

Example 3.1.4 *Consider*

$$\left\{\begin{array}{rl} (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2 = 2 & \textit{in} \quad \Omega = \{(x_1, x_2),\ x_1 > 0\}, \\ u(0, x_2) = x_2 & \textit{on} \quad \Gamma = \{(0, x_2),\ x_2 \in \mathbb{R}\}. \end{array}\right. \tag{3.1.13}$$

Here $E(x, z, p) = p_1^2 + p_2^2 - 2$. Therefore (3.1.2a)–(3.1.2c) becomes

$$\left\{\begin{array}{l} y_1' = 2p_1, \\ y_2' = 2p_2, \\ y(0) = (0, y_2^0), \quad y_2^0 \in \mathbb{R}, \end{array}\right. \qquad \left\{\begin{array}{l} z' = 4, \\ z(0) = y_2^0, \end{array}\right. \qquad \left\{\begin{array}{l} p_1' = 0, \\ p_2' = 0. \end{array}\right.$$

The general solution (y, z, p) can be written as

$$\left\{\begin{array}{rcl} y_1 &=& 2p_1^0 t, \\ y_2 &=& 2p_2^0 t + y_2^0, \end{array}\right. \qquad z = 4t + y_2^0, \qquad \left\{\begin{array}{rcl} p_1 &=& p_1^0, \\ p_2 &=& p_2^0. \end{array}\right.$$

Here y_2^0 and $p^0 = (p_1^0, p_2^0)$ are arbitrary constants. Using (3.1.13), which we assume holds in $\overline{\Omega}$, we can eliminate two of these three constants, for example p_1^0 and p_2^0, so that (y, z, p) depends only on y_2^0. Indeed, $(\partial_{x_1} u)^2 + (\partial_{x_2} u)^2 = 2$ in $\overline{\Omega}$ and $\partial_{x_2}(u - x_2) = 0$ on Γ give

$$\left\{\begin{array}{rcl} (p_1^0)^2 + (p_2^0)^2 &=& 2, \\ p_2^0 &=& 1, \end{array}\right. \quad \textit{so} \quad \left\{\begin{array}{rcl} p_1^0 &=& \pm 1, \\ p_2^0 &=& 1. \end{array}\right.$$

Hence, there are two solutions (y, z, p) given by

$$\left\{\begin{array}{rcl} y_1 &=& \pm 2t, \\ y_2 &=& 2t + y_2^0, \end{array}\right. \qquad z = 4t + y_2^0, \qquad \left\{\begin{array}{rcl} p_1 &=& \pm 1, \\ p_2 &=& 1. \end{array}\right.$$

Now, let $x = (x_1, x_2) \in \Omega$. We can find (t, y_1^0) such that $y(t, y_1^0) = x$. It follows $t = \pm\frac{1}{2}x_1$, $y_2^0 = x_2 \mp x_1$. Therefore, two classical solutions to 3.1.13 are given by

$$u(x_1, x_2) = u(y(t)) = z(t) = 4\left(\pm\frac{1}{2}x_1\right) + (x_2 \mp x_1) = x_2 \pm x_1.$$

3.2 Classical local solutions to first-order PDEs

In this section, we will construct *local classical solutions* to (3.0.1) by following the method of characteristics as described in Section 3.1. Note that in general the boundary Γ in (3.0.1b) is not flat, where, without loss of generality, we say that Γ is flat if

$$\Gamma \subset \{y^0 = (y_1^0, \ldots, y_{N-1}^0, 0),\ (y_1^0, \ldots, y_{N-1}^0) \in \mathbb{R}^{N-1}\}. \tag{3.2.1}$$

The following lemma shows that modulo a diffeomorphism, (3.0.1) is equivalent to a similar problem with a flat boundary. This motivates the study of (3.0.1) with Γ flat, because due to the simplicity of the boundary the analysis is easier. Next we will return to (3.0.1) with a non-flat boundary by using the following lemma again.

Lemma 3.2.1 *Consider the problem (3.0.1). Let $x^0 \in \Gamma$ and assume that Γ is of class C^{k+1} at x^0, i.e. there exist $G \subset \mathbb{R}^N$ open with $x^0 \in G$, and $\theta \in C^{k+1}(\overline{Q}, \overline{G})$ invertible, such that $\theta(Q^+) = \Omega \cap G =: G^+$, $\theta(Q^0) = \Gamma \cap G =: \Gamma_0$; see Definition 1.2.4.*

i) Assume that E is of class C^k, $g \in C^k(\overline{\Gamma_0})$ and $u \in C^k(\overline{G^+})$ solves (3.0.1) with $\Omega = G^+$ and $\Gamma = \Gamma_0$. If $\tilde{u}(\tilde{x}) = u \circ \theta^{-1}(\tilde{x})$, $\tilde{x} \in \overline{Q^+}$, then $\tilde{u} \in C^k(\overline{Q^+})$ and it solves

$$\tilde{E}(\tilde{x}, \tilde{v}, \nabla\tilde{v}) = 0 \quad \text{in } Q^+, \tag{3.2.2a}$$
$$\tilde{u}(\tilde{x}) = \tilde{g}(\tilde{x}) \quad \text{on } Q^0, \tag{3.2.2b}$$

with $\tilde{E}$ of class C^k and $\tilde{g} \in C^k(\overline{Q^0})$.

ii) Conversely, assume $\tilde{E}$ is of class C^k, $\tilde{g} \in C^k(\overline{Q^0})$, and $\tilde{u} \in C^k(\overline{Q^+})$ solves (3.2.2). If $u(x) = \tilde{u} \circ \theta^{-1}(x)$, $x \in \overline{G^+}$, then $u \in C^k(\overline{G^+})$ and it solves (3.0.1) with $\Omega = G^+$, $\Gamma = \Gamma_0$, E of class C^k, and $g \in C^k(\overline{\Gamma_0})$.

Proof. By the composition rule, it is clear that $\tilde{u} \in C^k(\overline{Q^+})$ and

$$\nabla u(x) = {}^t[\nabla\theta(\tilde{x})]^{-1} \cdot \nabla u(\tilde{x}), \quad \tilde{x} \in \overline{Q^+}, \ \ x = \theta(\tilde{x}) \in \overline{G^+}.$$

Therefore, (3.0.1) implies

$$\begin{aligned} 0 &= E(x, u(x), \nabla u(x)) = E(\theta^{-1}(\tilde{x}), \tilde{u}(\tilde{x}), {}^t[\nabla\theta(\tilde{x})]^{-1} \cdot \nabla\tilde{u}(\tilde{x})) \\ &=: \tilde{E}(\tilde{x}, \tilde{u}(\tilde{x}), \nabla\tilde{u}(\tilde{x})), \quad \tilde{x} \in Q^+, \end{aligned} \tag{3.2.3}$$

$$\begin{aligned} 0 &= u(x) - g(x) = \tilde{u}(\tilde{x}) - g(\theta(\tilde{x})) \\ &=: \tilde{u}(x) - \tilde{g}(\tilde{x}), \quad \tilde{x} \in Q^0, \end{aligned} \tag{3.2.4}$$

which proves i) with $\tilde{E}(\tilde{x}, \tilde{z}, \tilde{p}) = E(\theta^{-1}(\tilde{x}), \tilde{z}, {}^t[\nabla\theta(\tilde{x})]^{-1} \cdot \tilde{p})$ and $\tilde{g}(\tilde{x}) = g(\theta(\tilde{x}))$. For ii) we proceed similarly.

3.2.1 Classical local solutions: flat boundary

Under the assumptions of Lemma 3.2.1, the problem (3.0.1) is equivalent to the problem (3.2.2), which has a flat boundary. In this section, we assume that Γ in (3.0.1b) is flat and prove the existence of a classical solution to (3.0.1). In the following section we will show the existence of a classical solution to (3.0.1) in the case Γ is not flat, by using again Lemma 3.2.1.

3.2.1.1 Boundary and initial conditions

The equations (3.1.2) describe the evolution of u from (3.0.1) along the characteristic curves $\{y(t),\, t \in I\}$. We have to equip (3.1.2a)–(3.1.2c) with initial conditions; let's say

$$(y(0), z(0), p(0)) = (y^0, z_0, p^0). \tag{3.2.5}$$

It is worth emphasizing that z_0 and p^0 depend on y^0, so $z_0 = z_0(y^0)$, $p^0 = p^0(y^0)$. This implies that y, z, and p depend on (t, y^0), hence $y = y(t, y^0)$, $z = z(t, y^0)$, $p = p(t, y^0)$ (we will drop the variable y^0 whenever there is no confusion).

Clearly, the initial conditions for y and z follow naturally from the boundary condition $u = g$ on Γ, i.e.

$$y(0, y^0) = y^0, \quad z(0, y^0) = z_0(y^0) := g(y^0), \quad y^0 \in \Gamma. \tag{3.2.6}$$

Differentiating the equation for z in (3.2.6) with respect to y_i^0, $i = 1, \dots, N-1$, gives $\partial_i z_0 = \partial_i g$ on Γ, which leads to these conditions for p_i^0:

$$p_i^0(y^0) = \partial_i g(y^0), \quad i = 1, \dots, N-1, \quad y^0 \in \Gamma. \tag{3.2.7}$$

Condition (3.2.7) is called a "*compatibility condition*". Conditions (3.2.6) and (3.2.7) uniquely define all the initial conditions except p_N^0. For p_N^0 we assume that (3.0.1a) holds in $\Omega \cup \Gamma$, which implies the following equation counting for the remaining p_N^0:

$$E(y^0, z_0, p^0) = 0, \quad y^0 \in \Gamma. \tag{3.2.8}$$

A triple (y^0, z_0, p^0) satisfying (3.2.6)–(3.2.7) is called the "*admissible initial condition*".

Remember that we intend to find the solution u to (3.0.1) by starting at $y^0 \in \Gamma$, and then solving (y, z, p) along the characteristic curves, which will provide $u(y(t, y^0)) = z(y(t, y^0))$ along this curve. For this, the curves $y(t, y^0)$, which start on Γ, must evolve inside Ω. As the vector $y'(0, y^0)$ is tangent to the characteristic curve at y^0, this condition is satisfied, at least locally near Γ, if $y_N'(0, y^0) > 0$, which from (3.1.2a) is equivalent to

$$\partial_{p_N} E(y^0, z_0, p^0) > 0, \quad \forall y^0 \in \Gamma. \tag{3.2.9}$$

Note that this condition is equivalent to $\partial_{p_N} E(y^0, z_0, p^0) \neq 0$ because we can consider $-E$ instead of E. The condition (3.2.9) is called a "*transversality condition*". An initial

condition (y^0, z_0, p^0) satisfying (3.2.6)–(3.2.9) (so admissible and transversal) is called a "*non-characteristic*".[2]

The initial conditions $z_0 = z_0(y^0)$ and $(p^0_1, \dots, p^0_{N-1}) = (\partial_1 g(y^0), \dots, \partial_{N-1} g(y^0))$ are explicit and are given by (3.2.6) and (3.2.7), respectively. The remaining initial condition p^0_N is given implicitly by (3.2.8). The following result shows that if $(y^0, z_0(y^0), p^0(y^0))$ is non-characteristic at a certain x^0, then it can be extended uniquely as a non-characteristic initial condition near x^0.

Lemma 3.2.2 *Assume E and g are C^k, $k \geq 2$. Furthermore, assume that there exists $x^0 \in \Gamma$ and $p^0(x^0) \in \mathbb{R}^N$ such that (3.2.6)–(3.2.9) hold at $(x^0, z_0(x^0), p^0(x^0))$. Then there exist $\rho_0 > 0$ and a unique $p^0 \in C^{k-1}(\Gamma_0; \mathbb{R}^N)$ with $\Gamma_0 = B(x^0, \rho_0) \cap \Gamma$, such that (3.2.6)–(3.2.9) hold at $(y^0, z_0(y^0), p^0(y^0))$ for all $y^0 \in \Gamma_0$, i.e. $(y^0, z_0(y^0), p^0(y^0))$ is a non-characteristic initial condition for all $y^0 \in \Gamma_0$.*

Proof. Note that from (3.2.6) and (3.2.7) we have $z_0(y^0) = g(y^0)$, and $p^0_i(y^0) := \partial_i g(y^0)$, $i = 1, \dots, N-1$, are well-defined and of class C^{k-1} in a neighborhood of x^0. Now we will define p^0_N by using the (implicit function) Theorem 1.3.12. Consider

$$\begin{array}{llll} G: & (\mathbb{R}^{N-1} \times \{0\}) \times \mathbb{R} & \mapsto & \mathbb{R}, \\ & (y^0, p^0_N) & \mapsto & E(y^0, z_0(y^0), (p^0_1(y^0), \dots, p^0_{N-1}(y^0), p^0_N)). \end{array}$$

Note G is C^{k-1} near x^0, $G(x^0, p^0_N(x^0)) = 0$, and

$$\partial_{p_N} G(x^0, p^0_N(x^0)) = \partial_{p_N} E(x^0, z_0(x^0), p^0(x^0)) > 0.$$

From Theorem 1.3.12 there exists a unique $p^0_N \in C^{k-1}(\Gamma_0)$, with $\Gamma_0 = B(x^0, \rho_0) \cap \Gamma$ for a certain $\rho_0 > 0$, and $G(y^0, p^0_N(y^0)) = 0$ on Γ_0. As $\partial_{p_N} E(x^0, z_0(x^0), p^0(x^0)) > 0$, by continuity it follows that (3.2.9) holds in Γ_0 provided ρ_0 is small, which proves the lemma.

3.2.1.2 Classical local solutions

We have already seen that in order to solve (3.0.1) by using the method of characteristics, we are led to

$$\begin{cases} y' &= \nabla_p E(y, z, p), \\ z' &= p \cdot \nabla_p E(y, z, p), \\ p' &= -\nabla_x E(y, z, p) - p \partial_z E(y, z, p), \end{cases} \qquad \begin{cases} y(0, y^0) &= y^0, \\ z(0, y^0) &= z_0(y^0), \\ p(0, y^0) &= p^0(y^0), \end{cases} \tag{3.2.10}$$

with $y = y(t, y^0)$, $z = z(t, y^0)$, $p = p(t, y^0)$, and $y^0 \in \Gamma$, $z_0 = z_0(y^0)$, $p^0 = p^0(y^0)$ satisfying (3.2.6)–(3.2.9).

We have the following local existence theorem.

[2] The name refers to the fact that the boundary Γ is not tangent to the characteristic curves.

Theorem 3.2.3 *Assume that E and g are C^k functions, $k \geq 2$, and $(x^0, z_0(x^0), p^0(x^0))$ is non-characteristic for a certain $x^0 \in \Gamma$. Then there exist $\rho_0 > 0$ and $r_0 > 0$ such that the following hold.*

(i) There exists $y^0 \mapsto p^0(y^0) \in C^{k-1}(\Gamma_0; \mathbb{R}^N)$ such that $(y^0, z_0(y^0), p^0(y^0))$ is non-characteristic for all $y^0 \in \Gamma_0 := \Gamma \cap \overline{B}(x^0, \rho_0)$.

(ii) For every $y^0 \in \Gamma_0$, the initial value problem (3.2.10) has a unique $C^k([-r_0, r_0]; \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ solution. Furthermore, if T is defined by

$$\begin{array}{llll} T: & [0, r_0] \times \Gamma_0 =: U_0 & \mapsto & \Omega_0 := T(U_0) \subset \overline{\Omega}, \\ & (t, y^0) & \mapsto & T(t, y^0) := y(t, y^0), \end{array}$$

then T is C^{k-1}, invertible, and its inverse is C^{k-1}.

(iii) The function $u : \Omega_0 \mapsto \mathbb{R}$, $u(x) = z \circ T^{-1}(x) = z(y(t, y^0))$, is a $C^k(\overline{\Omega}_0)$ solution to

$$E(x, u, \nabla u) = 0 \ \text{ in } \ \Omega_0, \tag{3.2.11a}$$

$$u = g \ \text{ on } \ \Gamma_0. \tag{3.2.11b}$$

(iv) If $\partial_{p_N} E(x^0, z_0(x^0), (p_1^0(x^0), \ldots, p_{N-1}^0(x^0), q))$ does not change sign (remains positive or negative) for all $q \in \mathbb{R}$, then (3.2.11) has a unique C^k solution.

Proof. See Theorem 9.2.1 in Annex.

3.2.2 Classical local solutions: non-flat boundary

When the boundary Γ in (3.0.1b) is not flat, we proceed as follows. Given $x^0 \in \Gamma$, by using Lemma 3.2.1 we transform the problem (3.0.1) near x^0 to a similar problem with flat boundary, for which Theorem 3.2.3 provides a classical solution. By using Lemma 3.2.1 again we obtain a classical local solution to the problem (3.0.1). More specifically we have this result.

Theorem 3.2.4 *Let $x^0 \in \Gamma$ and assume Γ is C^{k+1} at x^0, $k \geq 2$, E and g are C^k, and the initial conditions $z_0(x^0)$, $p^0(x^0)$ satisfy*

$$\left\{ \begin{array}{ll} z_0(x^0) = g(x^0), & E(x^0, z_0(x^0), p^0(x^0)) = 0, \\ p_\tau^0(x^0) = \nabla_\tau g(x^0), & \nabla_p E(x^0, z_0(x^0), p^0(x^0)) \cdot \nu(x^0) > 0. \end{array} \right. \tag{3.2.12}$$

Here $\nu(x^0)$ is the unit normal vector to Γ at x^0 oriented inward Ω, p_τ^0 is the tangential component of p^0, and $\nabla_\tau g$ is the tangential gradient of g defined by

$$p_\tau^0 = p^0 - (p^0 \cdot \nu)\nu, \quad \nabla_\tau g = \nabla \mathbf{g} - (\nabla \mathbf{g} \cdot \nu)\nu, \tag{3.2.13}$$

with $\mathbf{g}$ being any C^k extension of g. Then there exists a solution $u \in C^k(\overline{\Omega} \cap \overline{B}(x^0, \rho_0))$ to (3.0.1), for a certain $\rho_0 > 0$ small. Furthermore, if

$$\nabla_p E(x^0, z_0(x^0), p_\tau^0(x^0) + q\nu(x^0)) \cdot \nu(x^0) \ \text{ does not change sign}, q \in \mathbb{R}, \tag{3.2.14}$$

then the solution is unique at least locally near x^0.

Proof. The proof is based on Lemma 3.2.1 and Theorem 3.2.3. As Γ is C^{k+1} at x^0, there exists $\theta \in C^{k+1}(\overline{Q};\overline{G})$ as in Lemma 3.2.1. We set $\tilde{x}^0 = \theta^{-1}(x^0)$, and consider the problem (3.2.2) with $\tilde{E}$, $\tilde{g}$, $\tilde{u}$ as in Lemma 3.2.1, which establishes a correspondence between the local solutions to (3.0.1) with non-flat boundary and (3.2.2) with flat boundary.
Claim: $\tilde{E}$, $\tilde{g}$ and $(\tilde{x}^0, \tilde{z}_0(\tilde{x}^0), \tilde{p}^0(\tilde{x}^0))$, where $\tilde{z}_0(\tilde{x}^0) = z_0(x^0)$, $\tilde{p}^0(\tilde{x}^0) = {}^t[\nabla\theta(\tilde{x}^0)] \cdot p^0(x^0)$, satisfy the conditions of Theorem 3.2.3 at $\tilde{x}^0$. Assuming the claim holds implies that (i)–(iii) of Theorem 3.2.3 hold, and the existence of a C^k solution u follows from Lemma 3.2.1.

Now we prove the claim. From (3.2.3) and (3.2.4), clearly $\tilde{E}$ and $\tilde{g}$ are C^k. It remains to show that $(\tilde{x}^0, \tilde{z}_0(\tilde{x}^0), \tilde{p}^0(\tilde{x}^0))$ is non-characteristic. First we note that the vectors $[\nabla\theta(\tilde{x}^0)]_{\cdot,j}$, $j = 1,\ldots,N-1$, are tangent to Γ at x^0 and the vector $[\nabla\theta(\tilde{x}^0)]_{\cdot,N}$ is oriented inside Ω. Combined with $[\nabla\theta(\tilde{x}^0)]^{-1} \cdot [\nabla\theta(\tilde{x}^0)] = Id$,[3] it implies that

$$\begin{aligned} ({}^t[\nabla\theta(\tilde{x}^0)] \cdot \nu(x^0))_i &= 0, \quad i = 1,\ldots,N-1, && (3.2.15)\\ [\nabla\theta(\tilde{x}^0)]^{-1}_{N,\cdot} &= \lambda\nu(x^0), \quad \lambda > 0. && (3.2.16) \end{aligned}$$

As every vector can be decomposed in its tangential and normal components, by using (3.2.15), for $i = 1,\ldots,N-1$ we get

$$\begin{aligned} \tilde{p}^0_i(\tilde{x}^0) &= ({}^t[\nabla\theta(\tilde{x}^0)] \cdot p^0(x^0))_i = ({}^t[\nabla\theta(\tilde{x}^0)] \cdot (p^0_\tau(x^0) + (p^0(x^0)\cdot\nu(x^0))\nu(x^0)))_i\\ &= ({}^t[\nabla\theta(\tilde{x}^0)] \cdot (p^0_\tau(x^0))_i = ({}^t[\nabla\theta(\tilde{x}^0)] \cdot \nabla_\tau g(x^0))_i\\ &= ({}^t[\nabla\theta(\tilde{x}^0)] \cdot \nabla\mathbf{g}(x^0))_i - ({}^t[\nabla\theta(\tilde{x}^0)] \cdot \nu(x^0))_i(\nabla\mathbf{g}(x^0)\cdot\nu(x^0))\\ &= ({}^t[\nabla\theta(\tilde{x}^0)] \cdot \nabla\mathbf{g}(x^0))_i = (\nabla(\mathbf{g}\circ\theta)(\tilde{x}^0))_i\\ &= \partial_i\tilde{g}(\tilde{x}^0). && (3.2.17) \end{aligned}$$

Also, from $E(x^0, z_0(x^0), p^0(x^0)) = 0$ and (3.2.3) we get

$$\begin{aligned} \tilde{E}(\tilde{x}^0, \tilde{z}_0(\tilde{x}^0), \tilde{p}^0(\tilde{x}^0)) &= E(\theta^{-1}(\tilde{x}^0), \tilde{z}_0(\tilde{x}^0), {}^t[\nabla\theta(\tilde{x}^0)]^{-1}\cdot\tilde{p}^0(\tilde{x}^0))\\ &= E(x^0, z_0(x^0), p^0(x^0)) = 0. && (3.2.18) \end{aligned}$$

Furthermore, using (3.2.16) and the first line of (3.2.18) gives

$$\begin{aligned} \partial_{\tilde{p}_N}\tilde{E}(\tilde{x}^0, \tilde{z}_0(\tilde{x}^0), \tilde{p}^0(\tilde{x}^0)) &= [\nabla\theta(\tilde{x}^0)]^{-1}_{N,\cdot} \cdot \nabla_p E(x^0, z_0(x^0), p^0(x^0)) && (3.2.19)\\ &= ([\nabla\theta(\tilde{x}^0)]^{-1}_{N,\cdot}\cdot\nu(x^0))(\nabla_p E(x^0, z_0(x^0), p^0(x^0))\cdot\nu(x^0)) > 0. \end{aligned}$$

From (3.2.17)–(3.2.19) it follows that $(\tilde{x}^0, \tilde{z}_0(\tilde{x}^0), \tilde{p}(\tilde{x}^0))$ is non-characteristic.
Claim: If furthermore (3.2.14) holds, the solution is unique. In view of Lemma 3.2.1, it is enough to show that the problem (3.2.2) has a unique solution. We show that Condition (iv) of Theorem 3.2.3 holds, which proves the claim. Indeed, first note that by using (3.2.16), for $\tilde{p}_N \in \mathbb{R}$ we have

$$ {}^t[\nabla\theta(\tilde{x}^0)]^{-1} \cdot (\tilde{p}^0_1(\tilde{x}^0), \ldots, \tilde{p}^0_{N-1}(\tilde{x}^0), \tilde{p}_N) $$

[3] $Id : \mathbb{R}^N \mapsto \mathbb{R}^N$ is the identity map.

$$\begin{aligned}
& {}^t[\nabla\theta(\tilde{x}^0)]^{-1}\cdot(\tilde{p}_1^0(\tilde{x}^0),\ldots,\tilde{p}_{N-1}^0(\tilde{x}^0),\tilde{p}_N^0(\tilde{x}^0)) + {}^t[\nabla\theta(\tilde{x}^0)]^{-1}\cdot(0,\ldots,0,\tilde{p}_N-\tilde{p}_N^0(\tilde{x}^0)) \\
= {} & p^0(x^0) + (\tilde{p}_N - \tilde{p}_N^0(\tilde{x}^0))[\nabla\theta(\tilde{x}^0)]^{-1}{}_{N,\cdot} \\
= {} & p^0(x^0) + \lambda(\tilde{p}_N - \tilde{p}_N^0(\tilde{x}^0))\nu(x^0) \\
= {} & p_\tau^0(x^0) + q\nu(x^0),
\end{aligned}$$

for a certain $q \in \mathbb{R}^N$, which by using (3.2.16) again implies that

$$\begin{aligned}
& \partial_{\tilde{p}_N}\tilde{E}(\tilde{x}^0,\tilde{z}_0(\tilde{x}^0),(\tilde{p}_1^0(\tilde{x}^0),\ldots,\tilde{p}_{N-1}^0(\tilde{x}^0),\tilde{p}_N)) \\
= {} & [\nabla\theta(\tilde{x}^0)]_{N,\cdot}^{-1}\cdot\nabla_p E(x^0,z_0(x^0),{}^t[\nabla\theta(\tilde{x}^0)]^{-1}\cdot(\tilde{p}_1^0(\tilde{x}^0),\ldots,\tilde{p}_{N-1}^0(\tilde{x}^0),\tilde{p}_N)) \\
= {} & \lambda(\nabla_p E(x^0,z_0(x^0),p_\tau^0(x^0)+q\nu(x^0))\cdot\nu(x^0))
\end{aligned}$$

does not change sign. Theorem 3.2.3 implies the uniqueness of the solution $\tilde{u}$, and Lemma 3.2.1 proves the uniqueness of the solution u. $\square$

The following example shows that, in general, if the condition (3.2.14) fails then the problem (3.0.1) does not have a unique solution.

Example 3.2.5 *Consider problem* (3.1.13), *where* $E(x,z,p) = p_1^2+p_2^2-2$, $g(x_1,x_2) = x_2$. *Hence for* $x^0 = (0,x_2^0) \in \Gamma$ *we have* $\nu(x^0) = (1,0)$, $z_0(x^0) = x_2^0$, $p_\tau^0(x^0) = \nabla g(x^0) - (\nabla g(x^0)\cdot\nu(x^0)\nu(x^0) = (0,1)$, *and*

$$\nabla_p E(x^0,z_0(x^0),\partial_\tau p^0(x^0)+q\nu(x^0))\cdot\nu(x^0) = \partial_{p_1}E(x^0,x_2^0,(q,1)) = 2q.$$

Hence, condition (3.2.14) *is not satisfied. On the other hand, we have seen in Example 3.1.4 that problem* (3.1.13) *has at least two solutions.*

The following example shows that if the transversality condition in (3.2.12) (or (3.2.9)) is not satisfied then, depending on g, (3.0.1) may not have a solution, or may have infinitely many solutions.

Example 3.2.6 *Consider*

$$\partial_1 u + \partial_2 u = 1 \;\; in \;\; \Omega := \{(x_1,x_2), \;\; x_1, \in \mathbb{R}, \; x_2 > x_1\}, \tag{3.2.20a}$$

$$u(x_1,x_1) = g(x_1) \;\; on \;\; \Gamma := \partial\Omega. \tag{3.2.20b}$$

Here $E = p_1 + p_2 - 1$ *and* $g(x_1)$ *is a smooth function. Note that for whatever choice of* p^0, *the transversality condition is not satisfied, because*

$$\nu = (-1,1)/\sqrt{2} \;\; and\, so \;\; \nabla_p E(y^0,z_0,p^0)\cdot\nu = 0.$$

So Theorem 3.2.3 does not apply. The system (3.2.10) *associated with* (3.2.20) *is*

$$\left\{\begin{array}{rclrclrcl}
y_1'(t,y_1^0) & = & 1, & y_2'(t,y_1^0) & = & 1, & z'(t,y_1^0) & = & 1, \\
y_1(0,y_1^0) & = & y_1^0, & y_2(0,y_1^0) & = & y_1^0, & z(0,y_1^0) & = & g(y_1^0),
\end{array}\right.$$

and their solution is

$$y_1(t, y_1^0) = t + y_1^0, \quad y_2(t, y_1^0) = t + y_1^0, \quad z(t, y_1^0) = t + g(y_1^0).$$

The characteristic curves are straight lines parallel to Γ*, so they do not intersect* Γ *unless* $y_1^0 = 0$*, in which case the characteristic curve coïncides with* Γ*. Therefore, we cannot find a solution to* (3.2.20) *by starting on* Γ *and following the characteristic curves.*
i) Case of no solution*. Note that if a classical solution* u *exists, if we set* $z(s) = u(x_1+s, x_2+s)$ *then* $z'(s) = 1$*, and therefore by integrating* $z'(s)$ *in* $(-x_1, 0)$ *we find that* u *must satisfy* $u(x_1, x_2) = u(0, x_2 - x_1) + x_1$*. For* $x_2 = x_1$ *we get* $u(x_1, x_1) = u(0, 0) + x_1 = g(0) + x_1$*. Hence, if* $g(x_1) \neq g(0) + x_1$ *for any* x_1*, then* (3.2.20) *has no solution.*
ii) Case of infinitely many solutions*. We assume that* $g(x_1) = C + x_1$ *for all* x_1 *and a certain* $C \in \mathbb{R}$*. We can find infinitely many solutions to* (3.2.20) *in the following way. We consider PDE* (3.2.20a) *but with a different boundary condition instead of* (3.2.20b)*. Namely, we choose a boundary* $\Sigma \subset \mathbb{R}^2$ *intersecting* Γ *at a certain* x^0*, such that the transversality condition on* Σ *is satisfied. Next we look for a solution* u *to* (3.2.20a) *equipped with a boundary condition* $u = h$ *on* Σ*, where* h *is such that* $h = g$ *on* $\Sigma \cap \Gamma$*, i.e.* $h(x^0) = g(x^0)$*.*

For example, we can take $\Sigma = \{(y_1^0, 0),\ y_1^0 \in \mathbb{R}\}$ *and* h *smooth such that* $h(0) = C$*, because here* $\Sigma \cap \Gamma = (0, 0)$*. So we consider*

$$\partial_1 u + \partial_2 u = 1 \ \text{ in } \ \Omega, \quad u(x_1, 0) = h(x_1), \quad x_1 \in \mathbb{R}. \tag{3.2.21}$$

Problem (3.2.21) *satisfies the conditions of Theorem 3.2.3. The solution to characteristic equations of* (3.2.21) *is given by*

$$y_1(t, y_1^0) = t + y_1^0, \quad y_2(t, y_1^0) = t, \quad z(t, y_1^0) = t + h(y_1^0).$$

For arbitrary $x = (x_1, x_2) \in \Omega$*, we look for* (t, y_1^0) *such that* $x = y(t, y^0)$*. It follows* $t = x_2$*,* $y_1^0 = x_1 - x_2$*, and then* $u(x_1, x_2) = x_2 + h(x_1 - x_2)$ *is a classical solution to* (3.2.21) *and to* (3.2.20)*, provided* h *is* C^1 *and* $h(0) = C$*. So for* $g(x_1) = C + x_1$*,* (3.2.20) *has infinitely many solutions as we can take for example* $h(x_1) = C + x_1^n$*,* $n \in \mathbb{N}$*.*

Theorems 3.2.3 and 3.2.4 guarantee only local classical solutions, which are obtained in the situation where the characteristic curves do not intersect. The following example shows that depending on the behavior of characteristic curves, we may have a local or global classical solution, or no classical solution at all.

Example 3.2.7 *Consider*

$$\partial_2 u + u\partial_1 u = \partial_2 u + \partial_1 \left(\frac{1}{2}u^2\right) = 0 \ \text{ in } \ \Omega = \{(x_1, x_2),\ x_2 > 0\}, \tag{3.2.22a}$$

$$u(x_1, 0) = g(x_1) \qquad \text{on } \ \Gamma = \partial\Omega, \tag{3.2.22b}$$

referred to as "Burgers' equation", which describes the dynamics of rarefied gases (viscous effects are neglected). The system (3.2.10) associated with (3.2.22) has the form

$$\begin{cases} y_1'(t,y_1^0) = z, & y_2'(t,y_1^0) = 1, \quad z'(t,y_1^0) = 0, \\ y_1(0,y_1^0) = y_1^0, & y_2(0,y_1^0) = 0, \quad z(0,y_1^0) = g(y_1^0), \end{cases}$$

which implies

$$y_1(t,y_1^0) = tg(y_1^0) + y_1^0, \quad y_2(t,y_1^0) = t, \quad z(t,y_1^0) = g(y_1^0).$$

So the characteristic curves $\{y(t,y_1^0) := (y_1(t,y_1^0), y_2(t,y_1^0)),\ t \in \mathbb{R}\}$ *are straight lines starting at* $(y_1^0, 0)$ *with slope* $g(y_1^0)$ *given equivalently by*

$$y_1 = y_2 g(y_1^0) + y_1^0,\ y_1^0 \in \mathbb{R}. \tag{3.2.23}$$

Given $x = (x_1, x_2)$, $x_2 > 0$, *we look for* (t, y_1^0) *such that* $y(t,y_1^0) = x$, *which gives*

$$t = x_2, \quad y_1^0 = x_1 - x_2 g(y_1^0). \tag{3.2.24}$$

As z *is constant along characteristic curves we get* $u(x) = g(y_1^0)$, *or equivalently*

$$u(x_1,x_2) = g(x_1 - x_2 u(x_1,x_2)). \tag{3.2.25}$$

The solution is implicit and its qualitative behavior depends on g.

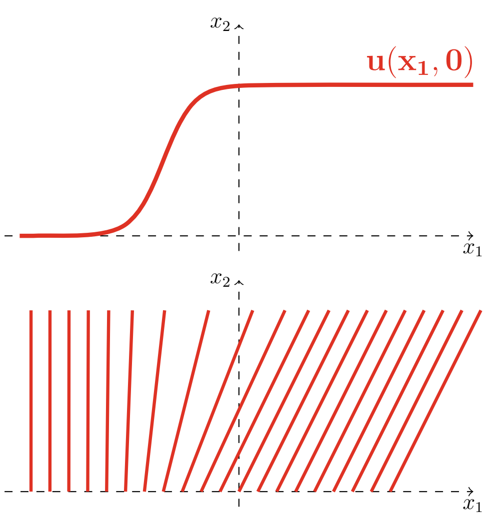

Figure 3.2.1: Rarefaction waves. Top: an increasing continuous (left) and discontinuous (right) initial condition. Bottom: the corresponding characteristic curves—they fill all the domain $\{x_2 > 0\}$ in the case of continuous initial condition (left), while there is a region where the characteristic curves do not cross (right).

Rarefaction waves. *Assume g is an increasing function (see Fig. 3.2.1, top). In this case the characteristic curves do not intersect. If g is continuous, see Fig. 3.2.1, top left, (3.2.22) has a solution in the entire domain, because the equation $y_1^0 = x_1 - x_2 g(y_1^0)$ has a unique solution y_1^0 for every (x_1, x_2). If g is discontinuous, see Fig. 3.2.1, top right, there is a region where there are no characteristic curves. In this region the method of characteristics does not provide a solution. Such a solution is called a "rarefaction wave". Clearly, if g is only continuous then u is only continuous. Furthermore, in the case g is discontinuous the method of characteristics does not provide a solution u defined globally; see Fig. 3.2.1, bottom, right.*

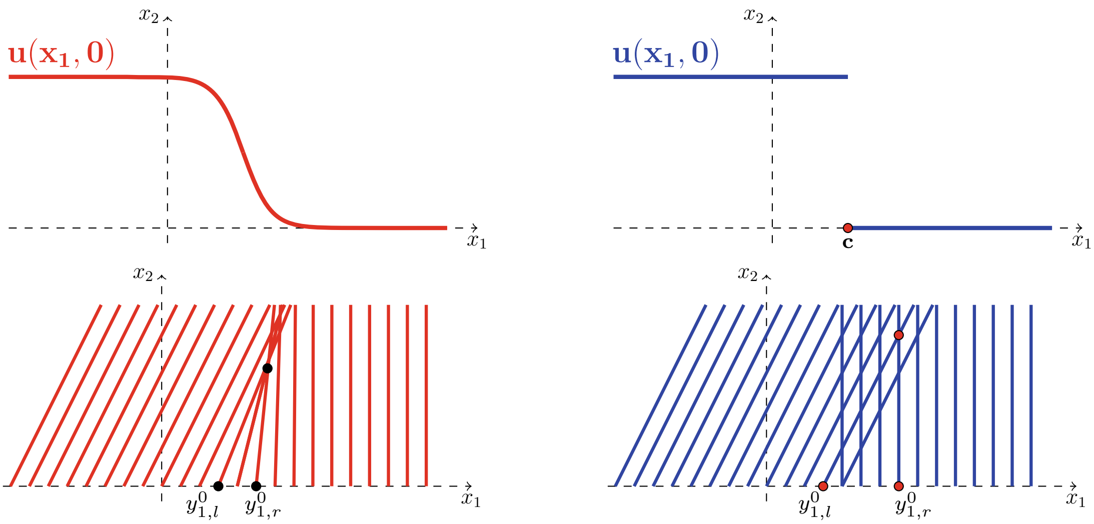

Figure 3.2.2: Shock waves Top: a decreasing continuous (left) and discontinuous (right) initial condition. Bottom: the corresponding characteristic curves, which in both cases intersect.

Shock waves. *Assume g is a decreasing function (see Fig. 3.2.2, top). The only difference with the case when g is increasing is that for every $x_2 > 0$, the equation $y_1^0 = x_1 - x_2 g(y_1^0)$ in (3.2.24), for certain (x_1, x_2), has two solutions $y_{1,l}^0$, $y_{1,r}^0$, $y_{1,l}^0 < y_{1,r}^0$. This implies that the two characteristic curves starting at $(y_{1,l}^0, 0)$ and $(y_{1,r}^0, 0)$ intersect. As z is constant along characteristic curves, z will attain the values $g(y_{1,l}^0)$ and $g(y_{1,r}^0)$ at the intersecting point (x_1, x_2). These values are in general different and therefore u cannot be continuous at the intersection point. Such solutions are called "shock waves". They are interesting, even though they are not continuous.*

For example, in the case $g = \mathbb{1}_{(-\infty,c)}$, for every (x_1, x_2) with $0 < x_1 - c < x_2$ we have $y_{1,l}^0 = x_1 - x_2$, $y_{1,r}^0 = x_1$. So u cannot be continuous at such (x_1, x_2) because $z = 1$ on the characteristic curve emanating from $y_{1,l}^0$ and $z = 0$ on the characteristic curve emanating from $y_{1,r}^0$. In particular, there are no classical solutions near $(c, 0)$. However, in the region $\{(x_1, x_2),\ x_1 < c,\ x_2 > \max\{0, x_1 - c\}$, resp. $\{(x_1, x_2),\ x_1 > c,\ 0 < x_2 < x_1 - c\}$, the problem has a unique classical solution $u = 1$, resp. $u = 0$.

3.3 Conservation laws and weak solutions

The examples we have seen so far in this chapter show that the method of characteristics, depending on the functions E, g, and on the boundary Γ, provides a local or global classical solution, which may be unique or not, or no classical solution at all. Of particular interest are the cases in which the classical local solutions interact and develop discontinuities, like in the *shock waves* case of Example 3.2.7.

We note that the method of characteristics provides the most natural way of constructing the solutions of (3.0.1). In this section we will discuss the concept of the weak solution to first-order PDEs, which extends the concept of classical solution obtained with the method of characteristics. We will also present the concept of *entropy solution*, which in the cases of non-uniqueness selects the solution that is physically admissible/reasonable.

We will restrict the discussion on the following class of PDEs:

$$\partial_2 u + \partial_1(f(u)) = 0, \qquad (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^+ := (-\infty, +\infty) \times (0, +\infty), \tag{3.3.1a}$$

$$u(x_1, 0) = g(x_1), \quad x_1 \in \mathbb{R}, \tag{3.3.1b}$$

where $u = u(x_1, x_2)$ is the unknown function, and g and f are given functions. The equation (3.3.1a) is a (scalar) conservation law in one space dimension and (3.3.1b) is the initial condition. Furthermore, u is the conserved quantity and f is its flux.

The name "*conservation law*" for (3.3.1) follows from this argument. Set $x = x_1$ (the space variable) and $t = x_2$ (the time), so (3.3.1) becomes

$$\partial_t u(x,t) + \partial_x(f(u(x,t)) = 0, \tag{3.3.2a}$$

$$u(x, 0) = g(x), \quad x \in \mathbb{R}. \tag{3.3.2b}$$

Assuming u is a smooth solution to (3.3.2), if $(a, b) \subset \mathbb{R}$ is a given fixed interval we get

$$\begin{aligned} \frac{d}{dt}\int_a^b u(x,t)dx &= \int_a^b \partial_t u(x,t)dx = -\int_a^b \partial_x(f(u(x,t))dx \\ &= f(u(a,t)) - f(u(b,t)). \end{aligned}$$

Hence, the total amount of u inside any given interval $[a, b]$ at any given time t remains constant, if for example the fluxes at the endpoints $f(u(a,t))$ and $f(u(b,t))$ are zero, or even if $f(u(a,t)) - f(u(b,t)) = 0$.

Remark 3.3.1 *The wave equation in one dimension*

$$\partial_{22} w - \partial_1(h(\partial_1 w)) = 0, \quad w = w(x_1, x_2)$$

can be written as follows. Let $u = (u_1, u_2) := (\partial_1 w, \partial_2 w)$. Then u solves the first-order vector PDEs

$$\partial_2 u + \partial_1 f(u) = 0, \quad f(u) = -(u_2, h(u_1)). \tag{3.3.3}$$

The equation (3.3.1) with $u = (u_1, \ldots, u_N)$ and $f(u) = (f_1(u), \ldots, f_N(u))$ (the equation (3.3.3) is an example with $N = 2$) is called "vector conservation laws in one space dimension". In order to avoid technicalities we will not consider vector conservation laws. The reader can read more on this topic in [16, 30–32].

Note that equation (3.3.1) is of the form (3.0.1a) with $E(x, z, p) = p_2 + f'(z)p_1$. About the classical solution to (3.3.1) we have this result, which is a straightforward corollary of Theorem 3.2.3.

Proposition 3.3.2 *Let $x^0 = (x_1^0, 0) \in \mathbb{R} \times \{0\}$ and assume f is C^{k+1} near x^0 in $\mathbb{R}^2$ and g is C^k near x_1^0 in $\mathbb{R}$, $k \geq 2$. Then*

$$(x^0, z_0(x^0), p^0(x^0)) = ((x_1^0, 0), g(x_1^0), (g'(x_1^0), -f'(g(x_1^0))g'(x_1^0)))$$

is a non-characteristic initial condition and $\partial_{p_2} E((x^0, z_0(x^0), p^0(x^0))) = 1$, and therefore there exists a unique C^k solution to (3.3.1) near x^0.

Note that the solution to the characteristic equations of (3.3.1) is given by

$$y_1(t, y_1^0) = f'(g(y_1^0))t + y_1^0, \quad y_2(t, y_1^0) = t, \quad z(t, y_1^0) = g(y_0^1), \quad y_1^0 \in \mathbb{R}. \tag{3.3.4}$$

We have seen this solution in Example 3.2.7, which dealt with a conservation law with $f(u) = \frac{1}{2}u^2$. As demonstrated in this example, if g is a decreasing function then the solutions along the characteristic curves create a discontinuity at the intersection of characteristic curves. The candidate solution obtained by the method of characteristic is discontinuous. The following definition extends the concept of the classical solution to a conservation law in order to allow a discontinuous solution.

Definition 3.3.3 *Assume $f \in L^1_{loc}(\mathbb{R})$ with $|f(z)| \leq C|z|$ for all $z \in \mathbb{R}$ and $C \geq 0$, and $g \in L^1_{loc}(\mathbb{R})$. A function $u \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^+)$ is said to be a "weak solution to (3.3.1)" if for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we have*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (u(x)\partial_2\varphi(x) + f(u)\partial_1\varphi(x))\, dx_1 dx_2 + \int_{\mathbb{R}} g(x_1)\varphi(x_1, 0) dx_1 \;=\; 0. \tag{3.3.5}$$

This definition is equivalent to say that (3.3.1) holds in the sense of distributions. It is obtained by multiplying (3.3.1) with the test function φ, integrating in $\mathbb{R} \times \mathbb{R}^+$ and finally using (3.3.1b). Note that the condition $|f(z)| \leq C|z|$ in Definition 3.3.3 is to ensure that $f(u) \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^+)$ which implies that (3.3.5) is well-defined.

In general, a weak solution to (3.3.1) is not a (classical) solution—we will see this in the following examples. The following proposition shows that if the weak solution is smooth then the concepts of weak and classical solutions coincide. The proof is elementary and can be found in many textbooks; see for example [16, 30, 43]. For the reader's convenience, we have included the proof in Theorem 9.2.2 in Annex.

Proposition 3.3.4 *Let $f \in C^1(\mathbb{R})$, $g \in C^0(\mathbb{R})$. If $u \in C^1(\mathbb{R}\times\mathbb{R}^+) \cap C^0(\mathbb{R}\times[0,\infty))$ is a classical solution to (3.3.1), then u is a weak solution to (3.3.1). Reciprocally, assume u is a weak solution to (3.3.1) and $u \in C^1(\Omega)\cap C^0(\overline{\Omega})$, where $\Omega \subset \mathbb{R}\times(0,\infty)$ is open. Then u is a classical solution to (3.3.1a) in Ω and satisfies $u = g$ on $\partial\Omega \cap \{x_2 = 0\}$.*

The following result, so-called "*Rankine-Hugoniot theorem*", essentially shows that the slope of the curve $x_1 = \chi(x_2)$, where the solution to (3.3.1) is discontinuous, equals the ratio of the jump of $f(u)$ over the jump of u on the discontinuity curve. The proof is simple and can be found, for example, in [16, 30, 43]. For the reader's convenience, we have given the proof in Theorem 9.2.3 in Annex.

Theorem 3.3.5 (Rankine-Hugoniot) *Let $\Omega \subset \mathbb{R}\times\mathbb{R}_+$ be a simply connected open set such that $\Omega = \Omega_l \cup \gamma \cup \Omega_r$, where γ is the graph of a function $x_1 = \chi(x_2)$, $x_2 \in (\alpha,\beta)$, $\gamma \in C^1(\alpha,\beta)$, and Ω_l, resp. Ω_r, is the open set at the left, resp. at the right, of γ; see Fig. 3.3.1. Assume that u is a weak solution to (3.3.1) in Ω such that $u \in C^1(\Omega_l)\cap C^0(\overline{\Omega}_l)$ and $u \in C^1(\Omega_r)\cap C^0(\overline{\Omega}_r)$, and set*

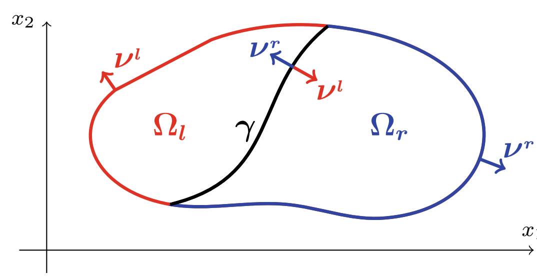

Figure 3.3.1: Ω_l, Ω_r and γ

$$\begin{aligned}
[u] &= \lim_{h\to 0^+}(u(\chi(x_2)+h,x_2) - u(\chi(x_2)-h,x_2)),\\
[f(u)] &= \lim_{h\to 0^+}(f(u(\chi(x_2)+h,x_2)) - f(u(\chi(x_2)-h,x_2)) \;\; x_2 \in (\alpha,\beta).
\end{aligned}$$

Then

$$\chi'[u] = [f(u)] \quad \text{on } \gamma. \tag{3.3.6}$$

Before we consider some examples, let us note that in physical situations x_2 is the time variable and x_1 is the space variable. So physically, χ' represent the speed of propagation of the discontinuity curve (chock speed). In the particular case of Burgers' equation (3.2.22), it gives

$$\chi' = \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{1}{2}\frac{u_r^2 - u_l^2}{u_r - u_l} = \frac{1}{2}(u_r + u_l).$$

Note that the characteristic curves on the left of the discontinuity curve have equations $y_1 = y_2 g(y_{1,l}^0) + y_{1,l}^0 = y_2 u_l + y_{1,l}^0$; see (3.2.23) and Example 3.2.7. Hence, $\dfrac{dy_1}{dy_2} = u_l$ is the propagation speed of the left characteristic curve (characteristic speed), and similarly u_r is the propagation speed of the right characteristic curve at a discontinuity point.

Therefore, the equation above for χ' states that the chock speed of Burgers' equation is the average of the characteristic speeds.

Burgers' equation (3.2.22) is a prototype of nonlinear first-order PDEs that, depending on the initial condition (3.2.22b), represents a number of features such as discontinuous weak solutions and non-uniqueness, as the following examples show.

Example 3.3.6 *We look for a weak solution to Burgers' equation* (3.2.22) *with*

$$g(x_1) = \mathbb{1}_{(-\infty,0)}(x_1). \tag{3.3.7}$$

Following Example 3.2.7, if a weak solution has a discontinuity curve γ, then from Rankine-Hugoniot Theorem 3.3.5 we get $\gamma = \{x_1 = \frac{1}{2}x_2,\ x_2 > 0\}$, because $[u] = -1$, $[f(u)] = -1/2$, and so $\chi' = \frac{[f(u)]}{[u]} = 1/2$. It implies that

$$u(x_1, x_2) = \begin{cases} 1, & x_1 < \frac{1}{2}x_2, \\ 0, & x_1 \geq \frac{1}{2}x_2 \end{cases} \tag{3.3.8}$$

is a candidate for a weak solution to (3.2.22)*. Let us show that u is a weak solution. For $\varphi \in \mathcal{D}(\mathbb{R}^2)$, let $R > 0$ such that $\operatorname{supp}(\varphi) \subset B_R$. Then we set $\Omega_l = B_R \cap \{(x_1, x_2),\ x_1 < \frac{1}{2}x_2,\ x_2 > 0\}$, $\Omega_r = B_R \cap \{(x_1, x_2),\ x_1 > \frac{1}{2}x_2,\ x_2 > 0\}$, and let ν^l, resp. ν^r, the exterior unit normal vector on $\partial\Omega_l$, resp. Ω_r. From the equation of γ, we find that $\nu^l = (2, -1)\frac{1}{\sqrt{5}}$, $\nu^r = -\nu^l$ on γ. Then using Gauss theorem*[4] *we get*

$$\begin{aligned}
& \int_{\mathbb{R}+} \int_{\mathbb{R}} (u\partial_2\varphi + f(u)\partial_1\varphi) dx_1 dx_2 + \int_{\{x_2=0\}} g(x_1)\varphi(x_1, 0) dx_1 \\
= & \int_{\Omega_l} (u_l\partial_2\varphi + f(u_l)\partial_1\varphi) dx + \int_{\partial\Omega_l \cap \{x_2=0\}} g(x_1)\varphi(x_1, 0) dx_1 \\
+ & \int_{\Omega_r} (u_r\partial_2\varphi + f(u)\partial_1\varphi dx + \int_{\partial\Omega_r \cap \{x_2=0\}} g(x_1)\varphi(x_1, 0) dx_1 \\
= & \int_{\partial\Omega_l \cap \{x_2=0\}} (u_l\nu_2^l + f(u_l)\nu_1^l)\varphi dx_1 + \int_{\partial\Omega_l \cap \{x_2=0\}} g(x_1)\varphi(x_1, 0) dx_1 \\
+ & \int_{\partial\Omega_r \cap \{x_2=0\}} (u_r\nu_2^r + f(u_r)\nu_1^r)\varphi dx_1 + \int_{\partial\Omega_r \cap \{x_2=0\}} g(x_1)\varphi(x_1, 0) dx_1 \\
- & \int_{\Omega_l \cup \Omega_r} (\partial_2 u + \partial_1(f(u)))\varphi dx - \int_\gamma ([u]\nu_2^l + [f(u)]\nu_1^l)\varphi ds \\
= & \ 0.
\end{aligned}$$

In general, conservation law equations (3.3.1) do not have a unique solution, as the following example shows.

[4] If $\Omega \subset \mathbb{R}^N$ is open and Lipschitz, $\nu = (\nu_1, \ldots, \nu_N)$ is the exterior unitary normal vector on $\partial\Omega$ and $f, g \in C^1(\overline{\Omega})$, then $\int_\Omega \partial_i f(x) g(x) dx = \int_{\partial\Omega} f(x) g(x) \nu_i ds - \int_\Omega f(x) \partial_i g(x) dx$.

Example 3.3.7 *Consider again Burgers' equation* (3.2.22) *with*

$$g(x_1) = \mathbb{1}_{(0,\infty)}(x_1), \tag{3.3.9}$$

as in Example 3.2.7. Note that the method of characteristics gives the solution

$$u(x_1,x_2) = \begin{cases} 0, & x_1 < 0, \\ 1, & x_1 \geq x_2, \end{cases} \tag{3.3.10}$$

leaving the solution undefined in $U = \{(x_1,x_2),\ 0 < x_1 < x_2\}$*; see Fig. 3.3.2. We can obtain many weak solutions by defining the solution in* U *in many ways, for example*

$$u(x_1,x_2) = \begin{cases} 0, & x_1 < 0, \\ x_1/x_2, & 0 \leq x_1 < x_2, \\ 1, & x_1 \geq x_2, \end{cases} \tag{3.3.11}$$

which gives a continuous solution, and

$$u(x_1,x_2) = \begin{cases} 0, & x_1 < \frac{1}{2}x_2, \\ 1, & x_1 \geq \frac{1}{2}x_2, \end{cases} \tag{3.3.12}$$

which gives a discontinuous solution; see Fig. 3.3.2.

There are several criteria to identify one among many solutions to a first-order conservation law, which is physically admissible/reasonable. We will consider only Lax entropy condition.

Definition 3.3.8 *Let* u *be a weak solution to the conservation law* (3.3.1) *that moreover is a piecewise classical solution, i.e.* u *is* C^1 *everywhere in* $\mathbb{R} \times \mathbb{R}^+$ *except on a set of smooth curves, where the solution is discontinuous but having side limits. We denote by* u_l*, resp.* u_r*, the restriction of* u *near a point on a discontinuity curve* $\gamma = \{(\chi(x_2), x_2),\ x_2 \in (a,b)\}$ *of* u *on the left, resp. right, of* C*. Such a solution* u *is said to be "admissible" if it satisfies the condition*

$$f'(u_l) > \chi' > f'(u_r), \tag{3.3.13}$$

at every point on the discontinuity curve γ*. This condition is called a "Lax entropy condition". A curve of discontinuity of* u *is called a "shock", if Lax entropy condition holds on the curve. A weak solution satisfying Lax entropy conditions is called a "Lax entropy solution".*

The motivation for the condition (3.3.13) is the idea that "*the information should always flow from the boundary onto the shock curve*"—so the values of u are transported from $\mathbb{R} \times \{0\}$, where u is known, to the chock curve. This is the same as saying that the characteristic curves start on $\mathbb{R} \times \{0\}$, where u is known, and end on the chock curve, which based on (3.3.4) is equivalent to (3.3.13). If the condition (3.3.13) is not satisfied then in general there are characteristic curves which start at the discontinuity curve and do not intersect the boundary—hence the information is "*created*" on this discontinuity curve, which is not physically admissible/reasonable.

Example 3.3.9 *Consider again Burgers' equation (3.2.22). If g is given by (3.3.7), see Fig. 3.2.2 right, then the solution given by (3.3.8) is an entropy solution because it satisfies Lax entropy condition on the discontinuity curve $\{x_1 = x_2/2\}$:*

$$f'(u_l) = 1 > \chi' = \frac{1}{2} > 0 = f'(u_r).$$

If g is given by (3.3.9), see Fig. 3.3.2, there are infinitely many solutions. The equation (3.3.11) gives a continuous solution, so it is not an entropy solution. The equation (3.3.12) gives a weak discontinuous solutions, however it is not an entropy solution because along its discontinuity curve $\gamma = \{(x_1, x_2),\ x_1 = \frac{1}{2}x_2,\ \ x_2 > 0\}$ it satisfies Rankine-Hugoniot condition but not Lax entropy condition:

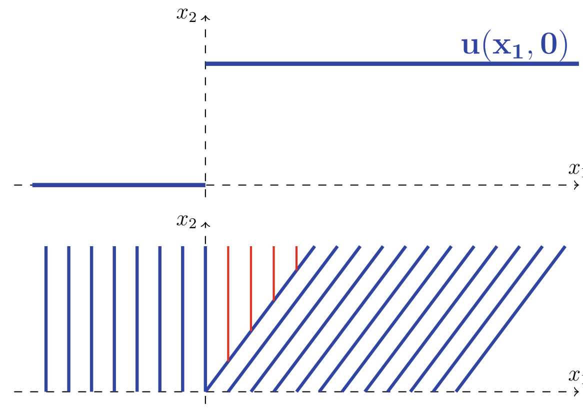

Figure 3.3.2: Some characteristic curves start at the discontinuity curves

$$\chi' = \frac{1}{2} \quad = \quad \frac{[f(u)]}{[u]}, \quad \textit{and} \quad f'(u_l) = 0 \not> \chi' \not> 1 = f'(u_r).$$

Here, we presented a short introduction to the first-order PDEs. We developed the concepts of characteristic equations and proved the existence of local classical solutions. We showed that, in general, a first-order PDE does not have a global classical solution, and developed the notion of the *"weak solution"*, which we demonstrated for single/scalar conservation laws. A first-order PDE may have many weak solutions, some of which may have discontinuities. A necessary condition for the weak solution along the discontinuity curves is given by the Rankine-Hugoniot condition. By using *Lax entropy condition*, we select one solution among many weak discontinuous solutions, which is physically admissible. There are other conditions which are used to select a physical reasonable solution among weak solutions, such as entropy condition or viscosity solutions.

Real physical problems involve systems of first-order multi-dimensional PDEs. The research on these PDEs is very active and the issues of existence and uniqueness are not completely solved. See, for example, [4, 10, 32] for more details on these topics.

Problems

Problem 3.1 *Let $b \in \mathbb{R}^N$, $c \in \mathbb{R}$ and f, resp. h, be a smooth function in $\mathbb{R}^N \times (0, \infty)$, resp. on $\mathbb{R}^N$, $u = u(x,t)$, and consider*

$$\begin{cases} u_t + b \cdot \nabla u + cu = f & \text{in} \quad \mathbb{R}^N \times (0, \infty), \\ \qquad\qquad\quad u = h & \text{on} \quad \mathbb{R}^N \times \{0\}. \end{cases}$$

Find a classical solution u by considering $\frac{d}{ds}u(x+sb,t+s)$.

Problem 3.2 *Find a non-trivial classical solution $u=u(x_1,x_2)$ to the following PDEs, by using the method of characteristics (here g is C^1).*

$$
\begin{array}{rrll rl r}
a) & x_1\partial_1 u - u\partial_2 u & = x_2 & \text{in } \{x_1>1\}, & u(1,x_2) & = x_2, & x_2\in\mathbb{R},\\
b) & u^2\partial_1 u + \partial_2 u & = 0 & \text{in } (0,\infty)^2, & u(x_1,0) & = \sqrt{x_1}, & x_1\in(0,\infty),\\
c) & x_1^2\partial_1 u + x_2^2\partial_2 u & = u^2 & \text{in } \{x_2>2x_1\}, & u(x_1,2x_1) & = x_1^2, & x_1\in\mathbb{R},\\
d) & \partial_1 u - \partial_2 u + (\partial_2 u)^2 & = 0 & \text{in } \mathbb{R}^2, & u(x_1,0) & = 0, & x_1>0,\\
e) & x_1\partial_1 u + x_2\partial_2 u + u & = 0 & \text{in } \mathbb{R}^2\backslash(0,0), & u(x_1,x_2) & = 1, & \{x_1^2+x_2^2=1\},\\
f) & (\partial_1 u)^2 + (\partial_2 u)^2 & = u^2 & \text{in } \mathbb{R}^2, & u(x_1,x_2) & = 1, & \{x_1^2+x_2^2=1\},\\
g) & \partial_1 u\partial_2 u & = u & \text{in } \{x_1>0\}, & u(0,x_2) & = x_2^2, & x_2\in\mathbb{R},\\
h) & \partial_1 u + \partial_2 u & = 1 & \text{in } \{x_2>0\}, & u(x_1,0) & = g(x_1), & x_1\in\mathbb{R},\\
i) & x_1\partial_1 u + (x_1+x_2)\partial_2 u & = 1 & \text{in } \{x_1>1\}, & u(1,x_2) & = g(x_2), & x_2\in\mathbb{R},\\
j) & x_1\partial_1 u + x_2\partial_2 u & = 2u & \text{in } \mathbb{R}\times(0,\infty), & u(x_1,1) & = g(x_1), & x_1\in\mathbb{R},\\
k) & (x_1+x_2)\partial_1 u + \partial_2 u & = 0 & \text{in } \mathbb{R}\times(0,\infty), & u(x_1,0) & = g(x_1), & x_1\in\mathbb{R},\\
l) & \partial_1 u + \partial_2 u & = u^2 & \text{in } \{x_2>0\}, & u(x_1,0) & = g(x_1), & x_1\in\mathbb{R}.
\end{array}
$$

Discuss the largest domain where u is defined and the uniqueness of the solution.

Problem 3.3 *Assume that g is an increasing C^1 function and consider*

$$u\partial_1 u + \partial_2 u + \frac{1}{2}u = 0 \ \text{ in } \{x_2>0\}, \quad u(x_1,0)=g(x_1).$$

i) Solve the characteristic equations to this PDE.
ii) Let $y(t,y_1^0)$ be the equations of the characteristic curve starting at $(y_1^0,0)$. Use the implicit function theorem to show that the map $(y_1^0,t)\mapsto(x_1,x_2)$, with $y(t,y_1^0)=(x_1,x_2)$ and $x_2>0$, is invertible and C^1.
iii) Conclude that there exists an implicit classical solution u.

Problem 3.4 *Consider the PDE*

$$(x_2^2+u)\partial_1 u + x_2\partial_2 u = 0, \qquad u=0 \ \text{ on } \ \Gamma=\left\{(x_1,x_2),\ \ 2x_1=x_2^2,\ x_2>1\right\}.$$

i) Show that every characteristic curve starting on Γ coïncides with Γ.
ii) Find a classical solution $u\neq 0$ in $\{x_2>1\}$.

Problem 3.5 *Consider the PDE*

$$x_2\partial_1 u - x_1\partial_2 u = 0 \quad \text{in } \mathbb{R}\times(0,\infty).$$

Verify whether the method of characteristics provides a classical solution to this PDE when equipped with one of the following boundary conditions:

i) $u(x_1,0)=x_1^2,\ \ x_1\in\mathbb{R}$, *ii)* $u(x_1,0)=x_1,\ \ x_1\in\mathbb{R}$, *iii)* $u(x_1,0)=x_1, x_1<0$.

Problem 3.6 *Consider the problem (3.3.1) with f and g as below. For each of them show that there exists an entropy solution.*[5]

a) $f(u) = \frac{1}{2}u^2, \quad g(x_1) = \mathrm{sign}(-x_1),$

b) $f(u) = \frac{2}{3}u^{\frac{3}{2}}, \quad g(x_1) = H(-x_1),$

c) $f(u) = u^3, \quad g(x_1) = e^{x_1}H(-x_1),$

d) $f(u) = \frac{1}{2}u^2, \quad g(x_1) = x_1\mathbb{1}_{(-\infty,0)} - \mathbb{1}_{(0,\infty)}.$

Problem 3.7 *Consider the problem (3.3.1) with $f \in C^1(\mathbb{R})$ non-decreasing and $f(0) = 0$, and assume that u is a weak piecewise solution with discontinuity along the curve $\gamma = \{(\chi(x_2), x_2), x_2 \geq 0\}$. Show that if f is convex, resp. concave, then shocks travel forward, resp. to the left, i.e. γ is an increasing, resp. decreasing, function.*

Problem 3.8 *Let $u : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$ be a piecewise smooth monotone decreasing in x_1 weak solution to (3.3.1) with a finite number of jumps. Show that if $f \in C^1(\mathbb{R})$ is a convex function then u satisfies the Lax entropy condition at each discontinuity point. Show also that if f is not convex then Lax entropy condition does not hold in general.*

[5] Hint: after solving the characteristic equations, you may "guess" the discontinuity curve γ by solving the differential equation $\chi' = \frac{[f(u)]}{[u]}$; next you find a solution candidate by using the method of characteristics and show that it is indeed an entropy solution.

4. Second-order linear elliptic PDEs: maximum principle and classical solutions

In this chapter, we will study the problem

$$-\Delta u = 0 \quad in \;\; \Omega, \tag{4.0.1a}$$
$$u = g \quad on \;\; \partial\Omega, \tag{4.0.1b}$$

called "*Dirichlet problem for the Laplacian*", as a prototype of second-order linear elliptic PDEs. A *classical solution* to the problem (4.0.1) is any function $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfying (4.0.1) pointwise.[1] We will prove the existence and uniqueness of a classical solution to this problem by using Perron's method. This method looks for the solution at the intersection of subsolutions and supersolutions to (4.0.1), and strongly uses the maximum principle, which roughly speaking states that the solution to a certain PDE in a domain Ω attains its maximum value on the boundary $\partial\Omega$. Note this method applies to more general second-order linear PDEs; see, for example, [20].

There are other methods which provide classical solutions, such as Schauder's theory, which in fact provides $C^{m,\alpha}(\overline{\Omega})$ solutions; see, for example, [20]. We note that, in general, (4.0.1) does not have a classical solution. This is a reason why the weak/variational solutions are introduced; see Chapter 7.

4.1 Laplace equation and the method of separation of variables

We start this chapter with the method of separation of variables (MSV) for (4.0.1). This elementary method, valid in Ω of rectangular form, provides a candidate solution in the form of a series. By using the mean value theorem and the maximum principle, we will show that this series provides a classical solution to (4.0.1).

[1] As we will see in this chapter, a classical solution u to (4.0.1) is C^∞, i.e. $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$.

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6_4

We consider the problem

$$-\Delta u = 0 \quad in \quad \Omega := (0,a)\times(0,b), \quad a,b>0, \tag{4.1.1a}$$
$$u = 0 \quad on \quad \partial\Omega\backslash\Sigma, \quad \Sigma := [0,a]\times\{b\}, \tag{4.1.1b}$$
$$u = g \quad on \quad \Sigma, \quad with \quad g\in C_0^0(\Sigma). \tag{4.1.1c}$$

Disregarding at this time the issues of existence and uniqueness of the solution (see Remark 4.1.1), we are interested only in finding a formula for the solution u. We assume that $u = u(x,y)$ can be written as $u(x,y) = V(x)W(y)$ (where the term *method of separation of variables* comes from). Replacing u in (4.1.1a) gives (Figure 4.1.1)

$$-(V''(x)W(y) + V(x)W''(y)) = 0. \tag{4.1.2}$$

Assuming $VW \neq 0$, from the last equation we obtain $\dfrac{V''(x)}{V(x)} = -\dfrac{W''(y)}{W(y)}$. Then, necessarily there exists a constant λ such that

$$\frac{V''(x)}{V(x)} = \lambda, \quad \frac{W''(y)}{W(y)} = -\lambda,$$

or equivalently

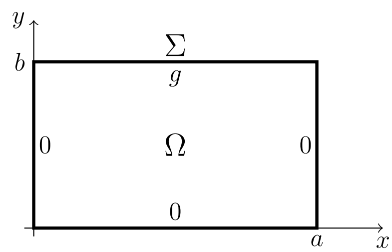

Figure 4.1.1: Domain, boundary, and boundary data

$$V'' - \lambda V = 0, \quad W'' + \lambda W = 0. \tag{4.1.3}$$

The differential equations (4.1.3) are of second order, linear, and homogeneous, so their solution spaces are of dimension two and their general solutions are

$$V(x) = Ax + B, \qquad W(y) = Cy + D, \qquad if \ \lambda = 0, \tag{4.1.4}$$
$$V(x) = Ae^{x\sqrt{\lambda}} + Be^{-x\sqrt{\lambda}}, \qquad W(y) = Ce^{y\sqrt{-\lambda}} + De^{-y\sqrt{-\lambda}}, \qquad if \ \lambda \neq 0, \tag{4.1.5}$$

with A, B, C, D being arbitrary constants. Here, $\sqrt{\pm\lambda}$ can be a complex number, and in this case A, B, C, and D must be such that the functions V and W are real. It follows that

$$u(x,y) = (Ax+B)(Cy+D), \qquad \lambda = 0, \tag{4.1.6}$$
$$u(x,y) = (Ae^{x\sqrt{\lambda}} + Be^{-x\sqrt{\lambda}})(Ce^{y\sqrt{-\lambda}} + De^{-y\sqrt{-\lambda}}), \qquad \lambda \neq 0, \tag{4.1.7}$$

is always a solution to (4.1.1a). The constants A, B, C, D are chosen such that u solves the boundary conditions (4.1.1b), (4.1.1c). Condition (4.1.1b) is written as

$$\left\{ \begin{array}{l} u(0,y) = u(a,y) = u(x,0) = 0, \\ \forall x \in (0,a), \quad \forall y \in (0,b), \end{array} \right. \iff V(0) = V(a) = W(0) = 0.$$

In the case $\lambda = 0$, by using (4.1.6) we get

$$B = Aa + B = D = 0,$$

which implies $A = B = D = 0$ and therefore $u = 0$. In the case $\lambda \neq 0$, by using (4.1.7) we get

$$A + B = Ae^{a\sqrt{\lambda}} + Be^{-a\sqrt{\lambda}} = C + D = 0.$$

Assuming $u \neq 0$, necessarily we get

$$0 \neq B = -A, \quad 0 \neq D = -C, \quad e^{2a\sqrt{\lambda}} = 1, \quad \sqrt{\lambda} = ik\frac{\pi}{a}, \quad \sqrt{-\lambda} = -k\frac{\pi}{a}, \quad k \in \mathbb{N}.$$

Therefore, $u(x, y)$ given by (4.1.7) can be written as

$$\begin{aligned} u_k(x, y) &= (-AD)(e^{ik\pi x/a} - e^{-ik\pi x/a})(e^{k\pi y/a} - e^{-k\pi y/a}) \\ &= (-4iAD)\sin(k\pi x/a)\sinh(k\pi y/a) \\ &=: \gamma_k \sin(k\pi x/a)\sinh(k\pi y/a). \end{aligned}$$

So, there are infinitely many $u_k(x, y)$ solving (4.1.1a) and (4.1.1b), and so formally

$$u(x, y) = \sum_{k\in\mathbb{N}} u_k(x, y) = \sum_{k\in\mathbb{N}} \gamma_k \sin(k\pi x/a)\sinh(k\pi y/a) \tag{4.1.8}$$

solves (4.1.1a) and (4.1.1b). Now, we address (4.1.1c). Replacing (4.1.8) in (4.1.1c) implies that the coefficients γ_k must satisfy

$$u(x, b) = \sum_{k\in\mathbb{N}} \gamma_k \sin(k\pi x/a)\sinh(k\pi b/a) = g(x), \quad x \in (0, a), \tag{4.1.9}$$

which is the sine Fourier expansion of g in $[0, a]$. Note that we have the following formulas[2]

$$\int_0^a \sin\left(k\frac{\pi}{a}x\right)\sin\left(l\frac{\pi}{a}x\right)dx = \delta_{kl}\frac{a}{2}, \quad k, l \in \mathbb{N}. \tag{4.1.10}$$

Then multiplying (4.1.9) by $\sin(l\pi x/a)$ and integrating in $(0, a)$ gives

$$\gamma_k = \frac{2}{a}\frac{1}{\sinh(k\pi b/a)}\int_0^a \sin(k\pi x/a)g(x)dx, \quad k \in \mathbb{N}.$$

Therefore, the solution u is given by

$$u(x, y) = \frac{2}{a}\sum_{k\in\mathbb{N}} \frac{\sin(k\pi x/a)\sinh(k\pi y/a)}{\sinh(k\pi b/a)}\int_0^a \sin(k\pi x/a)g(x)dx. \tag{4.1.11}$$

Finally, we note that we can use MSV for solving problems like

$$-\Delta u = 0 \;\; in \;\; \Omega = (0, a) \times (0, b), \quad u = g \;\; on \;\; \partial\Omega. \tag{4.1.12}$$

Indeed, if Σ_i, $i = 1, 2, 3, 4$ are the sides of $\partial\Omega$, then one can look for $u = u_1 + u_2 + u_3 + u_4$,

$$-\Delta u_i = 0 \;\; in \;\; \Omega = (0, a) \times (0, b), \quad u_i = g\mathbb{1}_{\Sigma_i} \;\; on \;\; \partial\Omega, \tag{4.1.13}$$

where $\mathbb{1}_{\Sigma_i}$ is the characteristic function of Σ_i. Each u_i can then be solved using MSV.

[2] Similarly, we have $\int_0^a \cos\left(k\frac{\pi}{a}x\right)\cos\left(l\frac{\pi}{a}x\right)dx = \delta_{kl}\frac{a}{2}$, $\int_0^a \sin\left(k\frac{\pi}{a}x\right)\cos\left(l\frac{\pi}{a}x\right)dx = 0$, $k, l \in \mathbb{N}$.

Remark 4.1.1 *Note that (4.1.11) is not found rigorously, and therefore we cannot say that the series (4.1.11) converges, nor that u is a classical solution to (4.1.1). To emphasize this, sometimes we will write $\sim$ instead of $=$ in (4.1.11).*

How can we verify whether the series in (4.1.11) provides a classical solution? The answer relies on the convergence of the Fourier series and the maximum principle. Indeed, if g is smooth then the sine Fourier series of g converges uniformly, which implies that the (4.1.11) converges in $C^0(\partial\Omega)$. Assuming this convergence holds, in combination with the maximum principle it implies the convergence of the series (4.1.11) in $C^0(\overline{\Omega})$, which in turn implies $u \in C^0(\overline{\Omega})$ and $u = g$ on $\partial\Omega$. Furthermore, by using the mean value theorem one proves that $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, which shows that u is a classical solution to (4.1.1). See Corollary 4.3.5 and examples 4.3.5, 4.3.6 for more details.

MSV, although useful, cannot be used when the domain is not a rectangle or a polar sector (see below). Theorem 4.4.8 provides general criteria for the solution to (4.0.1), which for the problem (4.1.1) implies that it has a unique classical solution if and only if $g \in C_0^0(\Sigma)$.

Remark 4.1.2 *MSV can be used to solve (4.1.1) with*

$$u = 0 \ \text{ on } \partial\Omega \backslash \Sigma, \quad \Sigma := (0, a) \times \{b\}, \tag{4.1.14a}$$

$$\partial_y u = g \ \text{ on } \Sigma. \tag{4.1.14b}$$

instead of (4.1.1b), (4.1.1c). One can proceed as above with the only difference that we define γ_k in (4.1.8) by using $\partial_y u(x, b) = g(x)$ instead of $u(x, b) = g(x)$.

Remark 4.1.3 *One can use MSV in polar coordinates, which is useful when the domain is a disk or a polar sector. For example, it is easy to show that if (r, θ) are polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, then the problem*

$$-\Delta u = 0 \quad \text{in } \Omega := B(0, R), \qquad u = g \quad \text{on } \partial\Omega,$$

can be written in polar coordinates as

$$-\left(\frac{1}{r}\partial_r\left(r\partial_r u\right) + \frac{1}{r^2}\partial_\theta^2 u\right) = 0, \quad (r, \theta) \in (0, R) \times [0, 2\pi), \tag{4.1.15a}$$

$$u(R, \theta) = g(\theta), \quad \theta \in [0, 2\pi). \tag{4.1.15b}$$

Then one looks for $u(r, \theta) = \mathcal{R}(r)\Theta(\theta)$, which after replacing in (4.1.15a) gives

$$\frac{1}{r}(r\mathcal{R}')'\Theta + \frac{1}{r^2}\mathcal{R}\Theta'' = 0.$$

Assuming $\mathcal{R}\Theta \neq 0$, dividing the equation above by $\mathcal{R}\Theta$ gives

$$\begin{cases} r^2\mathcal{R}'' + r\mathcal{R}' - \lambda\mathcal{R} = 0, & \Theta'' + \lambda\Theta = 0, \quad \Theta(0) = \Theta(2\pi) = 0, \\ r \in (0, R) & \theta \in (0, 2\pi), \qquad \Theta'(0) = \Theta'(2\pi), \end{cases} \tag{4.1.16}$$

with λ a constant. These equations have infinitely many solutions R_k, Θ_k, $k = 0, 1, \ldots$ given by

$$
\begin{aligned}
\mathcal{R}_0(r) &= C_0 + D_0 \ln r, & \Theta_0 = A_0, \\
\mathcal{R}_k(r) &= C_k r^k + D_k r^{-k},\ k = 1, 2, \ldots, & \Theta_k(\theta) = A_k \cos(k\theta) + B_k \sin(k\theta),\ k \in \mathbb{N}.
\end{aligned}
$$

Therefore, a series candidate solution to $-\Delta u = 0$ in $B(0, R)$ is given by

$$
u(r, \theta) \sim A_0(C_0 + D_0 \ln r) + \sum_{k=1}^{\infty} (C_k r^k + D_k r^{-k})(A_k \cos(k\theta) + B_k \sin(k\theta)).
$$

Assuming u is bounded implies $D_0 = D_1 = \cdots = 0$, and then

$$
u(r, \theta) \sim \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k (\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)). \tag{4.1.17}
$$

Finally, the coefficients α_k, β_k are defined formally by replacing the series (4.1.17) in (4.1.15b), multiplying by $\cos(l\theta)$, $\sin(l\theta)$, and integrating in $[0, 2\pi]$. It implies

$$
\alpha_0 = \frac{1}{\pi} \int_0^{2\pi} g(s) ds, \tag{4.1.18}
$$

$$
\alpha_k = \frac{1}{\pi} \int_0^{2\pi} g(s) \cos(ks) ds,\ \ \beta_k = \frac{1}{\pi} \int_0^{2\pi} g(s) \sin(ks) ds,\ \ k = 1, 2, \ldots. \tag{4.1.19}
$$

We conclude this remark by pointing out that we can use MSV to solve problems like

$$
-\Delta u = 0 \ \ in\ \ \Omega, \quad u = g \ \ on\ \ \partial\Omega,
$$

with $\Omega = \{(r, \theta),\ r \in [0, R),\ \theta \in (0, \sigma),\ \sigma \in (0, 2\pi)\}$, or $\Omega = \overline{B}(0, R)^c$.

4.2 Dirichlet problem in a ball

In this section, we will solve problem (4.0.1) in the case Ω is a ball. This problem is a building block of Perron's method that we will use for solving (4.0.1) with Ω general.

Theorem 4.2.1 (Dirichlet problem in a ball) *Let $R > 0$, $B_R := \{x \in \mathbb{R}^N,\ |x| < R\}$, V_N the volume of B_1, $g \in C^0(\partial B_R)$ and set*

$$
u(x) = \frac{R^2 - |x|^2}{N V_N R} \int_{\partial B_R} \frac{g(y)}{|x - y|^N} d\sigma(y), \quad x \in B_R. \tag{4.2.1}
$$

Then u is the unique $C^0(\overline{B}_R) \cap C^\infty(B_R)$ solution to (4.0.1) in $\Omega = B_R$.

The proof of this theorem is classical, see, for example, [20], and follows from direct calculations; see Theorem 9.3.1 in Annex for the details.

Remark 4.2.2 *Theorem 4.2.1 shows that if $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ is a classical solution to (4.0.1) then $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ because if $B \Subset \Omega$ is a ball and h solves $-\Delta h = 0$ in B, $h = u$ on ∂B, then $u = h$ so $u \in C^\infty(B)$, which implies $u \in C^\infty(\Omega)$ because $B \Subset \Omega$ is arbitrary.*

Remark 4.2.3 (Mean value theorem (MVT)) *The formula (4.2.1) is called "Poisson's formula". In an arbitrary ball $B_R(z) := B(z, R)$, it has the form*

$$u(x) = \frac{R^2 - |x-z|^2}{NV_N R} \int_{\partial B_R(z)} \frac{g(y)}{|x-y|^N} d\sigma(y), \quad x \in B_R(z), \tag{4.2.2}$$

which for $x = z$ gives

$$\begin{aligned} u(z) &= \frac{R^2}{NV_N R} \int_{\partial B_R(z)} \frac{g(y)}{R^N} d\sigma(y) = \frac{1}{|\partial B_R|} \int_{\partial B_R(z)} g(y) d\sigma(y) \\ &=: \fint_{\partial B_R(z)} u(y) d\sigma(y), \end{aligned} \tag{4.2.3}$$

where $|\partial B_R| = NV_N R^{N-1}$ is the area of ∂B_R.

Formula (4.2.3) is called a "mean value theorem in Ω": if $u \in C^2(\Omega)$ with $-\Delta u = 0$ in Ω, then $u(x)$, $x \in \Omega$, is equal to the mean value of u over the boundary of any ball $B(x, r)$, $B(x, r) \Subset \Omega$.

In the case $N = 2$, one can easily find the formula for $u(x)$ in (4.2.1) by using the method of separation of variables in polar coordinates; see Remark 4.1.3. Indeed, assume for simplicity $R = 1$. Then from (4.1.17), taking into account (4.1.19) and permuting the sum with the integrals gives

$$\begin{aligned} u(r,\theta) &\sim \frac{1}{\pi} \int_0^{2\pi} g(s) \left(\frac{1}{2} + \sum_{k=1}^{\infty} r^k (\cos(k\theta)\cos(ks) + \sin(k\theta)\sin(ks)) \right) ds \\ &= \frac{1}{\pi} \int_0^{2\pi} g(s) \left(\frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos(k(\theta - s)) \right) ds \\ &= \frac{1}{\pi} \int_0^{2\pi} g(s) \left(Re \left(\sum_{k=0}^{\infty} \left(re^{i(\theta-s)} \right)^k \right) - \frac{1}{2} \right) ds \\ &= \frac{1}{\pi} \int_0^{2\pi} g(s) \left(Re \left(\frac{1}{1 - re^{i(\theta-s)}} \right) - \frac{1}{2} \right) ds \\ &= \frac{1}{\pi} \int_0^{2\pi} g(s) \left(Re \left(\frac{1}{(1 - r\cos(\theta-s)) - ir\sin(\theta-s)} \right) - \frac{1}{2} \right) ds \\ &= \frac{1}{\pi} \int_0^{2\pi} g(s) \left(\frac{1 - r\cos(\theta-s)}{(1 - r\cos(\theta-s))^2 + (r\sin(\theta-s))^2} - \frac{1}{2} \right) ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} g(s)\frac{2(1-r\cos(\theta-s))-1+2\cos(\theta-s)-r^2}{1-2\cos(\theta-s)+r^2}ds \\
&= \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{g(s)}{1-2\cos(\theta-s)+r^2}ds \\
&= \frac{1-|x|^2}{2\pi}\int_{\partial B(0,1)} \frac{g(s)}{|x-y|^2}ds.
\end{aligned}
$$

The following proposition shows that the property (4.2.3), even though involving just an integral, which is well-defined for $u \in C^0(\Omega)$, actually is very strong.

Proposition 4.2.4 *Let $\Omega \subset \mathbb{R}^N$ be open and $u \in C^0(\Omega)$. The followings are equivalent:*

(i) $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω.

(ii) u satisfies the mean value theorem.

Proof. Indeed, (i) implies (ii) follows from Remark 4.2.3. Now let us prove that (ii) implies (i). Let $B = B(x,R) \Subset \Omega$, $h \in C^0(\overline{B}) \cap C^2(B)$ such that

$$h = u \ \text{ on } \ \partial B \quad \text{and} \quad -\Delta h = 0 \ \text{ in } \ B.$$

We will show that $u = h$ in B. Indeed, set $M = \max\{|u(y)-h(y)|,\ y \in \overline{B}\}$ and $K_M = \{y \in \overline{B},\ |u(y)-h(y)| = M\}$. Let us assume $M > 0$. As $u - h = 0$ on ∂B, there exists $x_M \in \partial K_M \cap B$ and $r_M > 0$ such that $B(x_M, r_M) \Subset B$, $|u(x_M) - h(x_M)| = M$, and $u - h$ does not change sign in B. From property (4.2.3) for u and h in B, we get

$$M = \pm(u(x_M) - h(x_M)) = \pm \fint_{\partial B(x_M,r_M)} (u(y)-h(y))d\sigma(y) < M,$$

with strict inequality holding because $u-h$ is continuous and $|u-h| \lneq M$ on $\partial B(x_M, r_M)$. The contradiction proves $M = 0$ and hence $u = h$ in B. From Theorem 4.2.1 and the arbitrariness of B, we get $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω.

4.3 Maximum principle for Laplacian

The proof of existence and uniqueness of solutions to the problem (4.0.1) for general domain Ω requires the maximum principle, subharmonic and superharmonic functions, and certain analysis results. We will prove the results we need for the operator $-\Delta$. Most of them hold for general second-order elliptic PDEs, and the related proofs are presented in section 9.3.2 in Annex.

All along this chapter, $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, unless otherwise specified, and

$$Lu := -\Delta u. \tag{4.3.1}$$

For the weak maximum principle, the reader can think of a function u in $[a,b] \subset \mathbb{R}$ with $-u''$ negative, resp. positive. Then u is concave up, resp. concave down, and therefore it reaches the maximum, resp. minimum, on the boundary of (a,b). The following theorem generalizes these facts in dimension N.

Theorem 4.3.1 (Weak maximum principle) *Let $\Omega \subset \mathbb{R}^N$ be open bounded and assume $Lu \le 0$, resp. $Lu \ge 0$, in Ω. Then the maximum, resp. minimum, of u in $\overline{\Omega}$ is achieved on $\partial\Omega$, i.e.*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \quad \textit{resp.} \quad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u,$$

where $\min\limits_A f = \min\{f(x),\ x \in A\}$, $\max\limits_A f = \max\{f(x),\ x \in A\}$ for every $f \in C^0(A)$.

Proof. We deal first with the case $Lu \le 0$. Let $x^0 \in \overline{\Omega}$ such that $u(x^0) = \max\{u(x),\ x \in \overline{\Omega}\}$. We distinguish two sub-cases.
(i) Case $Lu < 0$ in Ω. If $x^0 \in \partial\Omega$ the theorem is proved. So we assume $x^0 \in \Omega$. Hence $\nabla u(x^0) = 0$ and $\partial_{ii} u(x^0) \le 0$, for all i. Therefore $Lu(x^0) = -\Delta u(x^0) \ge 0$, which is a contradiction and proves that $x^0 \in \Omega$.
(ii) Case $Lu \le 0$ in Ω. Let $\epsilon > 0$ and consider $u_\epsilon(x) = u(x) + \epsilon e^{x_1}$, $x = (x_1, \dots, x_N)$. Then

$$Lu_\epsilon = Lu + \epsilon L e^{x_1} = Lu - \epsilon e^{x_1} < 0.$$

From Case (i), it follows that there exists $x^\epsilon = (x_1^\epsilon, \dots, x_N^\epsilon) \in \partial\Omega$ such that

$$u_\epsilon(x) \le u_\epsilon(x^\epsilon), \quad \textit{so} \quad u(x) \le u(x^\epsilon) + \epsilon(e^{x_1^\epsilon} - e^{x_1}), \quad \forall x \in \overline{\Omega}.$$

From the compactness of $\partial\Omega$ there exists a subsequence of x^ϵ, still denoted by x^ϵ, and $x^0 \in \partial\Omega$, such that $\lim_{\epsilon\to 0} x^\epsilon = x^0$, which implies that for all $x \in \overline{\Omega}$ we have $u(x) \le u(x^0) = \max\{u(x),\ x \in \overline{\Omega}\}$. So the claim for the case $Lu \le 0$ is proved.

For the case $Lu \ge 0$ we set $v = -u$, so $Lv \le 0$, and then the result follows from (ii). □

Corollary 4.3.2 (Comparison principle) *Let Ω be open bounded.*

(i) If $Lu \le Lv$ in Ω and $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω (monotonicity[3] of L).

(ii) If $Lu = Lv$ in Ω and $u = v$ on $\partial\Omega$, then $u = v$ in Ω (uniqueness of $Lu = 0$).

Proof. Set $w = u - v$.
(i) We have $Lw \le 0$ in Ω, $w \le 0$ on $\partial\Omega$. From Theorem 4.3.1 it follows $\max\limits_{\Omega} w \le \max\limits_{\partial\Omega} w \le 0$. So, $w \le 0$, or $u \le v$.

[3]This is the reason for the negative sign "$-$" in front of Δ.

(ii) As $Lw = 0$ and $w = 0$, (i) implies $0 \le w \le 0$, so $u = v$. $\square$

The result of Theorem 4.3.1 does not exclude that u achieves the maximum also at a certain point inside the domain. The strong maximum principle excludes this possibility. The following lemma and theorem are classical results; see, for example, [20].

Lemma 4.3.3 (Hopf lemma) *Let $\Omega \subset \mathbb{R}^N$ be an open set with $\partial\Omega$ a C^2 boundary near*[4] *$x_0 \in \partial\Omega$. Assume furthermore $Lu(x) \le 0$, $u(x) < u(x_0)$ for all $x \in \Omega$ near x_0 and u is differentiable at x_0. Then $\partial_\nu u(x_0) := \nabla u(x^0) \cdot \nu(x_0) > 0$, where $\nu(x_0)$ is the unit outward normal vector to $\partial\Omega$ at x^0.*

Proof. See Lemma 9.3.5 for the details of the proof. $\square$

In the case of a local minimum, we have the following result, which we still refer to as "*Hopf lemma*". Let $\Omega \subset \mathbb{R}^N$ be an open set with $\partial\Omega$ a C^2 boundary near $x_0 \in \partial\Omega$. Assume furthermore $Lu \ge 0$ in Ω, $u(x) > u(x_0)$ for all $x \in \Omega$ near x_0 and u is differentiable at x_0. Then $\partial_\nu u(x_0) < 0$. The proof follows directly from Lemma 4.3.3 applied to $-u$.

We note that under the assumptions of lemma it follows easily that $\partial_\nu u(x_0) \ge 0$. The strength of this lemma is that it rules out the case of equality. This result is strongly used in the following theorem (Figure 4.3.1).

Theorem 4.3.4 (Strong maximum principle) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and assume $Lu \le 0$, resp. $Lu \ge 0$. Then u is constant or, if not, u does not achieve its maximum, resp. minimum, in Ω.*

Proof. Let us consider first the case $Lu \le 0$. Let $M = \max\{u(x), x \in \overline{\Omega}$ and set $K = \{x \in \overline{\Omega}, u(x) = M\}$. Note that K is closed. Note also that $K \ne \emptyset$ because u attains its maximum M in $\overline{\Omega}$.

If $K = \overline{\Omega}$ then $u = M$ and the theorem is proved. Otherwise, we assume the theorem does not hold, i.e. u achieves its maximum in Ω. Hence, $K \cap \Omega \subsetneq \Omega$. Then there exists a ball $B_0 \Subset \Omega$, $B_0 \cap K = \emptyset$, and $x_0 \in \partial B_0$ such that u achieves at x_0 a strict maximum in B_0. Indeed, there exists $z \in \partial K \cap \Omega$ and $r > 0$ such that $B = B(z, r) \Subset \Omega$. The set $B \backslash K$ is open and nonempty. Let $B_0 = B(z_0, r_0)$, for a certain $z_0 \in B \backslash K$ and $r_0 > 0$, be the biggest open ball included in $B \backslash K$. From the maximality of B_0, necessarily $B_0 \cap \partial K \ne \emptyset$, and let $x_0 \in \partial B_0 \cap K$. It follows that $|\nabla u(x_0)| = 0$.

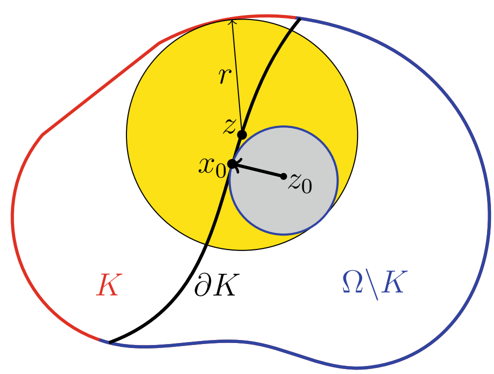

Figure 4.3.1: $B(z_0, r_0)$ and x_0.

[4]Here, "*$\partial\Omega$ a C^2 boundary near x_0*" means that in Definition 1.2.4 we replace "for every $x \in \partial\Omega$" with "for $x = x_0$". Note that this implies that $\Omega \cap B(x_0, r_0)$ lies on one side of $\partial\Omega$, for $r_0 > 0$ small.

Now we can apply Lemma 4.3.3 with B_0 instead of Ω. It gives $\partial_\nu u(x_0) > 0$, where $\nu = (x_0 - z_0)/|x_0 - z_0|$. But this contradicts the fact that $|\nabla u(x_0)| = 0$. Hence the theorem's claim holds.

The case $Lu \geq 0$ is proved by setting $v = -u$ and using the case $Lu \leq 0$. □

The following corollary is related to the solution to the problem (4.0.1) with the MSV; see Section 4.1. It shows that if the series representing the boundary function g convergences in $C^0(\partial\Omega)$ then the MSV provides a classical (unique) solution.

Corollary 4.3.5 *Let $\Omega \subset \mathbb{R}^N$ open bounded and $g, g_n \in C^0(\partial\Omega)$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} g_n = g$ in $C^0(\partial\Omega)$. Assume also that $u_n \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$, $n \in \mathbb{N}$, satisfies $\Delta u_n = 0$ in Ω, $u_n = g_n$ on $\partial\Omega$. Then there exists $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ such that $\lim_{n\to\infty} u_n = u$ in $C^0(\overline{\Omega})$ and*

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad u = g \quad \text{on} \quad \partial\Omega.$$

Proof. We note that (u_n) is a Cauchy sequence in $C^0(\overline{\Omega})$ because from

$$\Delta(u_n - u_m) = 0 \ \text{ in } \ \Omega, \quad u_n - u_m = g_n - g_m \ \text{ on } \ \partial\Omega,$$

and Theorem 4.3.1, we get $\|u_n - u_m\|_{C^0(\overline{\Omega})} \leq \|g_n - g_m\|_{C^0(\partial\Omega)}$. Therefore, $\lim_{n\to\infty} u_n = u$ in $C^0(\overline{\Omega})$ and $u = g$ on $\partial\Omega$, for a certain $u \in C^0(\overline{\Omega})$.

As $\Delta u_n = 0$ in Ω, we have $u_n(x) = ⨍_{\partial B_R(x)} u_n(y) d\sigma(y)$, for all $B_R(x) \Subset \Omega$. Passing to the limit in this equality implies $u(x) = ⨍_{\partial B_R(x)} u(y) d\sigma(y)$, which from Proposition 4.2.4, gives $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ and $\Delta u = 0$ in Ω. Then Theorem 4.2.1 implies $u \in C^\infty(\Omega)$.

Example 4.3.6 *Let us consider the problem*

$$\begin{aligned} \Delta u &= 0 \ \text{ in } \Omega = (0,\pi) \times (0,\pi), \\ u(x,y) &= g(x,y), \ \text{ with } \ g(x,y) = \begin{cases} 0, & (x,y) \in \partial\Omega \backslash \Sigma, \\ x(\pi - x), & (x,\pi) \in \Sigma = (0,\pi) \times \{\pi\}. \end{cases} \end{aligned}$$

a) Find a series solution candidate $u = u(x,y)$ of this problem.
b) Prove that the series converges to a solution $u \in C^1(\overline{\Omega}) \cap C^\infty(\Omega)$.
Solution.
i) We use the fact that the sine Fourier series of $x(\pi - x)$ is $\sum_{k=1}^\infty \frac{4}{\pi} \cdot \frac{1-(-1)^k}{k^3} \sin(kx)$, and that the convergence 5 is in $C^0([0,\pi])$.

Then using the MSV it is easy to find a series solution candidate

$$u(x,y) \sim \sum_{k=1}^\infty \frac{4}{\pi} \cdot \frac{1-(-1)^k}{k^3} \frac{\sinh(ky)}{\sinh(k\pi)} \sin(kx) =: \sum_{k=1}^\infty v_k(x,y).$$

ii.1) Set $u_n(x,y) = \sum_{k=1}^n v_k(x,y)$ in $\overline{\Omega}$ and $g_n = u_n$ on $\partial\Omega$. Then $\lim_{n\to\infty} \|g_n - g\|_{C^0(\partial\Omega)} = 0$. Indeed, $g_n = g = 0$ on $\partial\Omega\backslash\Sigma$, $g \in C^1(\overline{\Sigma}) \cap C^0_0(\overline{\Sigma})$ and from results

related to Fourier series[5] *we have* $\lim_{n\to\infty} \|g_n - g\|_{C^0(\overline{\Sigma})} = 0$.
ii.2) As $\Delta u_n = 0$ *in* Ω*, from Corollary 4.3.5 it follows that* $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ *and* $\Delta u = 0$ *in* Ω.
ii.3) In fact we even have $u \in C^1(\overline{\Omega})$*. Indeed, let* $\partial_i u_n = \sum_{k=1}^n \partial_i v_k$*, where* $i \in \{x, y\}$*. Then* $\partial_i u_n$ *is a Cauchy sequence in* $C^0(\overline{\Omega})$ *because*

$$|\partial_i v_k| \leq \frac{C}{k^2}, \quad C > 0.$$

Therefore, (u_n) *is a Cauchy sequence in* $C^1(\overline{\Omega})$ *and so* u_n *converges*[6] *to* u *in* $C^1(\overline{\Omega})$*, so* $u \in C^1(\overline{\Omega})$.

4.4 Solution to the Dirichlet problem

In this section, we will introduce the sub(super) harmonic functions and present some related results. We will also present some auxiliary results from analysis and conclude with the solution to the Dirichlet problem.

4.4.1 Sub(super) harmonic functions and sub(super) solutions

It turns out that the solution to (4.0.1) is at the intersection of two classes of functions, namely the subsolutions and supersolutions to (4.0.1). These sub(super) solutions have interesting properties listed below. We will show that the solution to (4.0.1) can be found as the supremum of subsolutions, or similarly as the infimum of supersolutions, to (4.0.1).

Definition 4.4.1 *Let* $\Omega \subset \mathbb{R}^N$ *open. We say* u *is subharmonic (superharmonic) in* Ω *if*

i) $u \in C^0(\overline{\Omega})$ *and*

ii) for every ball $B \Subset \Omega$*, for every* $h \in C^0(\overline{B}) \cap C^2(B)$ *with* $-\Delta h = 0$ *in* B *we have*

$$u \leq h \ (u \geq h) \text{ on } \partial B \quad \text{implies} \quad u \leq h \ (u \geq h) \text{ in } B.$$

We say u *is a subsolution (supersolution) to Dirichlet problem (4.0.1) if* u *is subharmonic (superharmonic) in* Ω *and* $u \leq g$ $(u \geq g)$ *on* $\partial\Omega$.

The concept of subsolution (supersolution) is trivially extended for operators L like (9.3.5), by just replacing $-\Delta$ by L in Definition 4.4.1. Below, we list a number of properties of subharmonic (superharmonic) functions. For their proof, see section 9.3.3 in Annex, or [20].

[5] If $f \in C^0_0([a, b]) \cap Lip(a, b)$ then the sine Fourier series of f converges in $C^0([a, b])$ to f.
[6] This is the so-called "*Weierstrass M-test*", which says: given a set $\Omega \subset \mathbb{R}^N$ and a series of functions $\sum_{k=1}^\infty u_k(x)$, with $|u_k(x)| \leq M_k$ for all $x \in \Omega$, where $M_k \geq 0$ and $\sum_{k=1}^\infty M_k < \infty$, then $\sum_{k=1}^\infty |u_k(x)|$ converges in $C^0(\overline{\Omega})$.

Proposition 4.4.2 *Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that $-\Delta u = 0$ in Ω.*

i) u is a subharmonic and a superharmonic function.

ii) Assume u solves (4.0.1). Then u is a subsolution and supersolution to (4.0.1).

Proposition 4.4.3 *Let $u, v \in C^0(\overline{\Omega})$.*

i) Assume u is subharmonic and v is superharmonic in Ω with $u \leq v$ on $\partial\Omega$. Then either $u < v$ or $u = v$ in Ω.

ii) Assume u is a subsolution and v is a supersolution to (4.0.1). Then either $u < v$ or $u = v$ in Ω.

Proposition 4.4.4 *Let u be subharmonic in Ω, B a ball with $B \Subset \Omega$, and $\overline{u} \in C^0(\overline{B}) \cap C^2(B)$ such that $-\Delta \overline{u} = 0$ in B and $\overline{u} = u$ on ∂B. Then U given by*

$$U(x) = \overline{u}(x), \;\; x \in B, \qquad U(x) = u(x), \;\; x \in \Omega \backslash B,$$

called "harmonic lifting of u with respect to B", is also subharmonic in Ω.

Proposition 4.4.5 *If $u_1, \ldots, u_m$ are subharmonic functions in Ω then $\max\{u_1, \ldots, u_m\}$ is also subharmonic in Ω.*

It follows that the solution u to (4.0.1) belongs to the intersection of subsolutions and supersolution to (4.0.1). Furthermore, one may look for it as the supremum of all subsolutions to (4.0.1), which is the idea behind Perron's method that we will present in section 4.4.2.

The following result is key to showing that the supremum of subsolutions to (4.0.1) is a harmonic function. For its proof, see Theorem 9.3.15 in Annex.

Theorem 4.4.6 *Let $B \subset \mathbb{R}^N$ be a ball and (u_n) a sequence of uniformly bounded and harmonic functions in B. Then (u_n) has a subsequence converging pointwise in B to a harmonic function u in B. Furthermore, the convergence is in $C^0(K)$ for every compact $K \subset B$.*

4.4.2 Solution to the Dirichlet problem

Now we state the main result of this section.

Theorem 4.4.7 (Perron) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded C^2 domain. Then for every $g \in C^0(\partial\Omega)$ there exists a unique $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ solution to (4.0.1).*

Comments on the proof of the theorem. The proof is classical and can be found in several books; see, for example, [20, 43]. For the reader's convenience, we have included it in section 9.3.3.3 in Annex. Here we give some comments which highlight the main steps and technique of the proof.

The proof considers separately the problem in the interior, $-\Delta u = 0$ in Ω, and the problem on the boundary, $u = g$ on $\partial\Omega$.

For the problem in the interior, one looks for the solution u of the form

$$u = \sup\{v,\ v \textit{ subsolution of } (4.0.1)\}.$$

By heavily using the maximum principle, it is proved that $-\Delta u = 0$ in Ω.

For the problem on the boundary, one uses the maximum principle again, and a so-called "*barrier function*" w at $\xi \in \partial\Omega$, defined by

$$\begin{aligned} w(x) &= \ln \frac{|x-y|}{R}, && \textit{for} \quad N = 2, \\ w(x) &= R^{2-N} - |x-y|^{2-N}, && \textit{for} \quad N \geq 3, \end{aligned}$$

with y and R such that $B(y,R) \subset \Omega^c$ and $\partial B(y,R) \cap \partial\Omega = \{\xi\}$. Note that w satisfies

$$w \in C^\infty(\mathbb{R}^N \backslash \{y\}), \quad -\Delta w = 0 \ \textit{ in } \ \Omega, \quad w > 0 \ \textit{ in } \ \overline{\Omega}\backslash\{\xi\}, \quad w(\xi) = 0. \tag{4.4.1}$$

Therefore w (resp. $-w$) is a supersolution (resp. subsolution) to the problem

$$-\Delta u = 0 \ \textit{in } \Omega,\ u = 0 \ \textit{on } \partial\Omega.$$

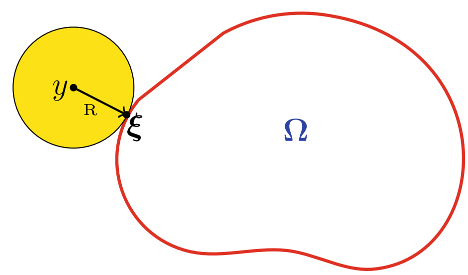

Figure 4.4.1: Ω, ξ, and $B(y,R)$

Any function w satisfying (4.4.1) is called a "*barrier function (for $-\Delta$) at ξ relative to Ω*". Comparing the function $u(x)$ with $g(\xi) \pm (\epsilon + kw(x))$, with $\epsilon > 0$ arbitrary, and $k > 0$ chosen appropriately, one proves

$$-(\epsilon + kw(x)) \leq u(x) - g(\xi) \leq (\epsilon + kw(x)), \quad \forall x \in \overline{\Omega},$$

which shows that $u = g$ on $\partial\Omega$. □

One may wonder whether the barrier function w is just a tool used in the proof of Theorem 4.4.7, or if it has a fundamental role in the existence of the solution to (4.0.1). It is also important to know whether the C^2 regularity of Ω can be weakened. Note that the C^2 regularity of Ω is used when constructing the barrier function w (namely when choosing the ball $B(y,R)$) (Figure 4.4.1).

A careful look at the proof of Theorem 4.4.7 shows that instead of the definition (4.4.1) for the barrier function, one can define a *weak barrier function* w *(for* $-\Delta$*) at* ξ *relative to* Ω as follows:

$$\begin{cases} (i) & w \textit{ is superharmonic in } \Omega \cap B(\xi, R_\xi), \\ (ii) & w > 0 \ \textit{ in } \Omega \cap B(\xi, R_\xi)\backslash\{\xi\}, \quad w(\xi) = 0, \end{cases} \tag{4.4.2}$$

for a certain $R_\xi > 0$. Then the proof of Theorem 4.4.7 remains unchanged.

In this context, we say that a point $\xi \in \partial\Omega$ is *regular with respect to the Laplacian* if there exists a (weak) barrier function (for $-\Delta$) at ξ relative to Ω.

The following theorem explains the connection between barrier functions and the problem (4.0.1); see [20].

Theorem 4.4.8 *Let $\Omega \subset \mathbb{R}^N$ be open, bounded. The Dirichlet problem (4.0.1) has a unique solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ for arbitrary $g \in C^0(\partial\Omega)$ if and only if all $\xi \in \partial\Omega$ are regular w.r.t. the Laplacian (i.e. for every $\xi \in \partial\Omega$ there exists a weak barrier function for $-\Delta$ at ξ relative to Ω).*

Note that the only regularity for Ω needed in Theorem 4.4.8 is the existence of a weak barrier function at every $\xi \in \partial\Omega$. This is a much weaker condition than the assumption "Ω *is of class* C^2". However, characterizing the domains that have a weak barrier function at each point of their boundary is not an easy problem, especially in high dimension. In dimension $N = 2$ one can show that if $\xi \in \partial\Omega$ then

$$w(x) = w(r, \theta) = -\mathrm{Re}\frac{1}{\ln z} = -\frac{\ln r}{\theta^2 + \ln^2 r},$$

with (r, θ) the polar coordinates, $x = \xi + (r\cos\theta, r\sin\theta)$, $z = re^{i\theta}$, are a weak barrier at ξ for $-\Delta$ relative to Ω, provided $\ln z$ is well-defined near every ξ on $\partial\Omega$. Hence, when $N = 2$ the problem (4.0.1) is solvable in Ω for arbitrary $g \in C^0(\partial\Omega)$, if for example each point $\xi \in \partial\Omega$ is the endpoint of a simple arc lying in the exterior of Ω because in this case one can define a branch of $\ln z$ near every $\xi \in \partial\Omega$.

In dimension $N \geq 3$, the characterization of Ω such that (4.0.1) is solvable is more difficult. An example given by Lebesgue, see [9], shows that in dimension $N = 3$ there are domains with non-regular points, such that (4.0.1) is not solvable for all $g \in C^0(\partial\Omega)$.

We conclude this chapter by emphasizing that we proved the existence and uniqueness of (4.0.1) using the method of subsolutions and supersolutions due to Perron. It provides *classical solutions*, i.e. $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. This method separates the problem in the interior from that on the boundary, and can be applied to general linear second-order elliptic PDEs. It is based heavily on the maximum principle.

Another approach to classical solutions is Schauder's theory, which is a complete and attractive PDE theory that provides $C^{2,\alpha}$ solutions to linear second-order elliptic PDEs, if both Ω and boundary data are of class $C^{2,\alpha}$; see [20] for details.

At first look, one would expect that problem (4.0.1) has a classical solution for all Ω bounded and open, and $g \in C^0(\partial\Omega)$. But as we mentioned above, this is not true in general (Lebesgue's example; see [9]). Furthermore, there are problems such as

$$-\Delta u = 0, \qquad \textit{in } \Omega = (-1,1)\times(0,1), \tag{4.4.3a}$$
$$u = \ln|1-\ln|x_1||, \qquad \textit{on } \Sigma = (-1,1)\times\{0\}, \tag{4.4.3b}$$
$$u = 0, \qquad \textit{on } \partial\Omega\backslash\Sigma, \tag{4.4.3c}$$

which do not have classical solutions but have "weak solutions" (we will review this problem in Chapter 7). The method of weak solutions has been developed to provide solutions to (4.0.1) in the cases where the method of classical solutions fails; see chapter 7.

Problems

Problem 4.1 *Consider the problem*

$$\begin{aligned} -\Delta u &= f \ \textit{in } \Omega = B(0,1) \subset \mathbb{R}^2, \\ u &= g \ \textit{on } \partial\Omega. \end{aligned}$$

i) Let $(r,\theta) \in (0,\infty)\times[0,2\pi)$ be the polar coordinates, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Show that $\Delta u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u$.

ii) Assume u, f, and g are smooth radially symmetric functions and solve for u.

Problem 4.2 *Show that $u(x) = \ln|x|$ is a classical solution to $\Delta u = 0$ in $\mathbb{R}^2\backslash\{(0,0)\}$.*

Problem 4.3 *For each of the following problems, find a series solution candidate and analyze the existence of a classical solution as required.*

i)
$$\begin{aligned} u_{xx} + u_{yy} = 0 \quad &\textit{in} \quad \Omega = (0,1)\times(0,2), \\ u = 0 \quad &\textit{on} \quad \partial\Omega\backslash\Sigma, \\ u = x(1-x) \quad &\textit{on} \quad \Sigma = (0,1)\times\{2\}. \end{aligned}$$

Show the existence of a classical solution $u \in C^1(\overline{\Omega}) \cap C^\infty(\Omega)$.

ii)
$$\begin{aligned} u_{xx} + u_{yy} = 0 \quad &\textit{in} \quad \Omega = (0,2)\times(0,1), \\ u = 0 \quad &\textit{on} \quad \partial\Omega\backslash\Sigma, \\ u = x\mathbb{1}_{(0,1)} + (1-x)\mathbb{1}_{(1,2)} \quad &\textit{on} \quad \Sigma = (0,2)\times\{1\}. \end{aligned}$$

Problem 4.4 *For each of the following problems, find a series solution candidate and analyze the existence of a classical solution as required.*

i)
$$\begin{aligned} u_{xx}+u_{yy}&=0 \quad \text{in} \quad \Omega=(0,\pi)\times(0,\pi),\\ \partial_\nu u&=0 \quad \text{on} \quad \partial\Omega\backslash\Sigma,\\ \partial_\nu u&=y-a \quad \text{on} \quad \Sigma=\{\pi\}\times(0,\pi), \quad a=\tfrac{\pi}{2}. \end{aligned}$$

Show the existence of a classical solution $u\in C^1(\overline{\Omega})\cap C^\infty(\Omega)$.

ii)
$$\begin{aligned} u_{xx}+u_{yy}&=0 \quad \text{in} \quad \Omega=(0,\pi)\times(0,\pi),\\ \partial_\nu u&=0 \quad \text{on} \quad \partial\Omega\backslash\Sigma,\\ \partial_\nu u(2,y)&=x^2-a \quad \text{on} \quad \Sigma=(0,\pi)\times\{\pi\}, \quad a=\tfrac{\pi^2}{3}. \end{aligned}$$

Show the existence of a classical solution $u\in C^1(\overline{\Omega})\cap C^\infty(\Omega)$.

In these problems, ν *is the unitary exterior normal vector on* $\partial\Omega$*, excluding the corners of the domain.*

Problem 4.5 *Let* $\alpha\in(0,\pi)$, $\Omega=\{(r,\theta),\ r\in(0,1),\ \theta\in(0,\alpha)\}$, $\gamma_0=\{(r,0),\ r\in(0,1)\}$, $\gamma=\{(1,\theta),\ \theta\in(0,\alpha)\}$ *and* $\gamma_\alpha=\{(r,\alpha),\ r\in(0,1)\}$. *Consider the problem*

$$-\Delta u=0 \ \text{ in } \Omega, \quad u=0 \ \text{ on } \gamma_0\cup\gamma_\alpha, \quad u=g \ \text{ on } \gamma,$$

with $g(1,\theta)=\theta(\alpha-\theta)$. *Use the method of separation of variables to find a series classical solution* $u\in C^0(\overline{\Omega})\cap C^2(\Omega)$.

Problem 4.6 *Prove that the problem*

$$-\Delta u=0 \ \text{ in } \Omega:=\{x\in\mathbb{R}^2,\ 0<|x|<1\}, \quad u=1 \ \text{ on } \{|x|=1\}, \quad u(0)=0,$$

does not have a $C^0(\overline{\Omega})\cap C^2(\Omega)$ *solution.*

Problem 4.7 *Prove that the problem*

$$-\Delta u=1 \ \text{ in } \Omega:=\{x\in\mathbb{R}^2,\ 0<|x|<1\}, \quad u=0 \ \text{ on } \partial\Omega,$$

does not have a $C^0(\overline{\Omega})\cap C^2(\Omega)$ *solution.*

Problem 4.8 *Let* $\Omega\subset\mathbb{R}^N$ *be a* C^2 *bounded open connected set,* ν *the unit outward normal vector to* $\partial\Omega$*, and* $u\in C^1(\overline{\Omega})\cap C^2(\Omega)$ *satisfying*

$$-\Delta u=0 \ \text{ in } \Omega, \quad \partial_\nu u=0 \ \text{ on } \partial\Omega.$$

Prove that u *is constant in* Ω.

Problem 4.9 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in C^0(\overline{\Omega})$. Prove that u is subharmonic in Ω if and only if*

$$u(x) \leq \frac{1}{NV_N R^{N-1}} \int_{\partial B_R(x)} u d\sigma =: \fint_{\partial B_R(x)} u d\sigma,$$

for all $x \in \Omega$ and $R > 0$ such that $B_R(x) \Subset \Omega$.

Problem 4.10 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Prove that $-\Delta u \leq 0$ in Ω if and only if*

$$u(x) \leq \fint_{\partial B_R(x)} u d\sigma,$$

for all $x \in \Omega$ and $R > 0$ such that $B_R(x) \Subset \Omega$.

Problem 4.11 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Prove that u is subharmonic in Ω if and only if $-\Delta u \leq 0$ in Ω.*

Problem 4.12 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Prove that $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ and $-\Delta u = 0$ in Ω if and only if $u \in C^0(\overline{\Omega})$ and satisfies the MVT in Ω.*

Problem 4.13 *Show that MVT in $\Omega \subset \mathbb{R}^N$ is equivalent to*

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) dy, \quad \forall B(x,r) \Subset \Omega.$$

Problem 4.14 *Prove the so-called "Liouville's theorem", i.e. if $u \in C^2(\mathbb{R}^N)$ satisfies*

$$-\Delta u = 0 \ \text{ in } \mathbb{R}^N, \quad |u| \leq M < \infty,$$

for a certain $M > 0$, then u is constant.

Problem 4.15 *Assume $\Omega \subset \mathbb{R}^N$ is an open bounded set and $0 \neq u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ solving*

$$-\Delta u = \lambda u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega \quad (\lambda \in \mathbb{R}).$$

Prove that $\lambda > 0$.

Problem 4.16 *Let $\Omega = \{(r,\theta),\ r \in [0,1),\ \theta \in (0,\alpha),\ \alpha \in (0,\pi)\}$ and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be the solution to*

$$-\Delta u = 0 \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega,$$

where $g \in C^0(\partial\Omega)$ with $g(r,0) = g(r,\alpha) = 0$, $r \in [0,1]$.

i) Assume that $|g(1,\theta)| \le C \sin\left(\frac{\pi}{\alpha}\theta\right)$*, for all* $\theta \in [0,\alpha]$*. Use the comparison principle to show that the* u *is differentiable at* $(0,0)$ *and* $\nabla u(0,0) = (0,0)$*.*

ii) Prove that the claim in (a) holds even with the condition for g *replaced by* $g(1,\theta) \in C^1([0,\alpha])$*.*

Problem 4.17 *Assume* $\Omega_+ \subset \mathbb{R}^2_+ := \{(x_1,x_2),\ x_2 > 0\}$ *is an open bounded set with* $\Gamma_0 = [0,a] \times \{0\} = \partial\Omega_+ \cap \partial\mathbb{R}^2_+$ *and*

$$u_+ \in C^0(\overline{\Omega}_+) \cap C^2(\Omega_+), \quad \text{with} \quad -\Delta u_+ = 0 \ \text{ in } \ \Omega_+, \qquad u_+ = 0 \ \text{ on } \ \Gamma_0.$$

Set $\Omega_- = \{(x_1,x_2) \ \text{s.th.}\ (x_1,-x_2) \in \Omega_+\}$ *and*

$$u_- \in C^0(\overline{\Omega}_-) \cap C^2(\Omega_-), \quad u_-(x_1,x_2) := -u_+(x_1,-x_2).$$

Set furthermore $\Omega = \Omega_+ \cup \Gamma_0 \cup \Omega_-$ *and* $u : \Omega \mapsto \mathbb{R}$ *with* $u = u_+$ *in* Ω_+ *and* $u = u_-$ *in* Ω_-*. Prove*[7,8] *that* $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *and* $-\Delta u = 0$ *in* Ω*.*

Problem 4.18 *Assume* $\Omega \subset \mathbb{R}^2$ *is a polygon with nodes* x^n*,* $n = 1,\dots,m$*, and consider the problem*

$$-\Delta u = 1 \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega.$$

i) Prove that this problem has a unique solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$*.*

ii) Prove that $u \in C^\infty(\overline{\Omega}\backslash\{x_1,\dots,x_m\})$*.*

iii) Prove that if Ω *is a convex polygon then* u *is differentiable at* x^n *and* $|\nabla u(x^n)| = 0$*, for all* n*.*

Problem 4.19 *Assume* $\Omega \subset \mathbb{R}^2$ *is a regular polygon with nodes* x^n*,* $n = 1,\dots,m$*, and consider the problem*

$$-\Delta u = 1 \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega.$$

Prove that $u \in C^\infty(\overline{\Omega}\backslash\{x_1,\dots,x_m\})$ *and there exist*[9] $M > 0$ *and points* $y_1,\dots,y_m$ *on* $\partial\Omega$ *such that* $|\partial_\nu u(y_i)| \ge M$*, for all* i*.*

Problem 4.20 *Assume* $\Omega \subset \mathbb{R}^N$ *is a* C^2 *bounded open set and* $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ *solves*

$$-\Delta u = 0 \ \text{ in } \ \Omega, \qquad u = g \ \text{ on } \ \Gamma_D \subset \partial\Omega, \qquad \partial_\nu u + cu = 0 \ \text{ on } \ \Gamma_N = \partial\Omega\backslash\Gamma_D,$$

with $\Gamma_D \ne \emptyset$ *and* $c > 0$*. Prove that* $\|u\|_{C^0(\overline{\Omega})} \le \|g\|_{C^0(\Gamma_D)}$*.*

[7] You may use Proposition 4.2.4.

[8] This is the so-called "*Schwarz reflection principle*".

[9] You may compare u with $v \in C^0(\overline{B}) \cap C^2(B)$ solving $-\Delta v = 1$ in B and $v = 0$ on ∂B, where B in the ball inscribed to the polygon.

5. Distributions

Well before Laurent SCHWARTZ developed the theory of distributions, see [47], physicists and engineers used distributions in an informal way, for example when solving ordinary differential equations with discontinuous right-hand sides.

Distributions generalize the concept of a function and as such they can represent very irregular "*functions*". At the same time, distributions are "*very regular*", as they do have derivatives of any order. It turns out that these features are powerful tools, because they allow us to elegantly write and solve differential and partial differential equations, which otherwise would not even have meaning.

The literature on distributions is rich, and we refer the reader to [18, 22, 23, 47, 50].

5.1 Motivation

Consider the initial value problem $y_h' = f_h$ in $\mathbb{R}$, $y_h(-\infty) := \lim\limits_{t\to\infty} y_h(t) = 0$, where

$$f_h(t) = \begin{cases} \dfrac{1}{h}, & t \in (a, a+h), \\ 0, & t \notin (a, a+h), \end{cases} \qquad \textit{with} \;\; a \in \mathbb{R},\; h > 0.$$

We look for the limit of y_h as $h \to 0$. Clearly, as $y_h(t) = \int_{-\infty}^{t} f(\tau)d\tau$ we get $y(t) = \lim_{h\to 0} y_h(t) = H(t-a)$, $t \neq a$, where the limit exists pointwise a.e. and in $L^1_{loc}(\mathbb{R})$, and $H(t)$ is the Heaviside function, $H(t) = \mathbb{1}_{(0,\infty)}(t)$.

One would prefer to consider the limit from a functional viewpoint—first consider the limit of f_h and then by using the differential equation deduce the limit for y_h. In this case, we note that $\lim\limits_{h\to 0} f_h(t) = 0$ for all t. If naively we would pass to the limit in $y_h' = f_h$, we would get $y' = 0$, so $y = 0$, which is incorrect. If one would accept that $\delta_a := \lim\limits_{h\to 0} f_h$ exists, let us say in $L^1(\mathbb{R})$, then necessarily $\delta_a = 0$. However, this is impossible because, as for arbitrary $h > 0$ we have $\int_{\mathbb{R}} f_h(t)dt = 1$, we obtain $\int_{\mathbb{R}} \delta_a(t)dt = 1$, which contradicts $\delta_a = 0$. So, if the limit δ_a of (f_h) exists, it cannot be in $L^1(\mathbb{R})$.

One notes that for $\varphi \in C_0^0(\mathbb{R})$, we have

$$\lim_{h\to 0} \int_{\mathbb{R}} f_h(t)\varphi(t)dt = \varphi(a).$$

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6_5

This equality serves as a definition for the limit of f_h as $h \to 0$. By definition, δ_a is a distribution, commonly called "*Dirac measure (or Dirac function) at a*". It belongs to $(C_0^0(\Omega))'$ and is characterized by

$$\langle \delta_a, \varphi \rangle_{(C_0^0(\Omega))' \times C_0^0(\Omega)} = \varphi(a), \quad \forall \varphi \in C_0^0(\mathbb{R}).$$

Here and afterwards, $\langle T, \varphi \rangle_{V' \times V}$ denotes a pairing/duality between $T \in V'$ and $\varphi \in V$, where V is a vector space and V' its dual space. Whenever there is no ambiguity, we will drop the symbol $V' \times V$ from $\langle T, \varphi \rangle_{V' \times V}$.

So we expect the limit y of y_h to satisfy $y' = \delta_a$, which is to be understood in the distribution sense, i.e. $\langle y', \varphi \rangle = \varphi(0)$, for all $\varphi \in \mathcal{D}(\mathbb{R})$. This simple example motivates not only the question of convergence of f_h, but also of y_h, y_h', and the distributional differential equation $y' = f$. These questions will be answered in a more general context with the results we will present in this chapter.

5.2 Distributions

In this section, we will introduce the space of test functions as well as the definition of distributions and of their derivatives, and we will provide a number of examples.

5.2.1 Test functions

We will introduce the class of test functions together with an associated topology, and some properties related to them.

Definition 5.2.1 *Let $\Omega \subset \mathbb{R}^N$ be a nonempty open set. A function $\varphi \in \mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$[1] defined in (1.1.28) is called a "test function in Ω", and $\mathcal{D}(\Omega)$ is called "the space of test functions in Ω". Sometimes we will write $\mathcal{D}$ instead of $\mathcal{D}(\mathbb{R}^N)$.*

For $\varphi \in \mathcal{D}(\Omega)$ we define $\operatorname{supp}(\varphi) := \overline{\{x \in \Omega,\ \varphi(x) \neq 0\}}$, the closure of the set $\{x \in \Omega,\ \varphi(x) \neq 0\}$, which is compact. We call it "support of φ".

Given a compact set $K \subset \Omega$ we define $\mathcal{D}_K(\Omega) = \{\varphi \in \mathcal{D}(\Omega),\ \ \operatorname{supp}(\varphi) \subset K\}$.

Clearly, the set $\mathcal{D}(\Omega)$ is a linear space. For the needs of this introduction, it is enough to define in $\mathcal{D}(\Omega)$ only the convergence of sequences.

Definition 5.2.2 *Let φ, φ_n, $n \in \mathbb{N}$ be elements of $\mathcal{D}(\Omega)$. We say "φ_n converges to φ in $\mathcal{D}(\Omega)$" and write "$\lim\limits_{n\to\infty} \varphi_n = \varphi$ in $\mathcal{D}(\Omega)$", if*

(i) $\exists K \subset \Omega$ with K compact, $\operatorname{supp}(\varphi) \subset K$, $\operatorname{supp}(\varphi_n) \subset K$ for all $n \in \mathbb{N}$, and

(ii) $\forall \alpha \in \mathbb{N}_0^N$, $\lim\limits_{n\to 0} \|D^\alpha \varphi_n - D^\alpha \varphi\|_{C^0(K)} = 0$.

[1] All along the remainder of this book, unless otherwise specified, the functions of $\mathcal{D}(\Omega)$ are with values in $\mathbb{C}$.

We note that $\mathcal{D}(\Omega)$ is not empty. For example, if ρ is the function defined in Example 1.1.13, then $\rho(n(x-x_0))$ belongs to $\mathcal{D}(\Omega)$ provided $x_0 \in \Omega$ and $n \in \mathbb{N}$ is big enough.

Remark 5.2.3 *Let $\varphi \in \mathcal{D}(\Omega)$ and (φ_n) be a sequence in $\mathcal{D}(\Omega)$ converging to φ in $\mathcal{D}(\Omega)$. Then from Definition 5.2.2 it follows that $(D^\alpha \varphi_n)$ converges to $D^\alpha \varphi$ in $\mathcal{D}(\Omega)$, for arbitrary $\alpha \in \mathbb{N}_0^N$.*

Using the sequence (ρ_n) of Example 1.1.13 and the "*convolution*" operator, we can construct infinitely many $\mathcal{D}(\Omega)$ functions.

Theorem 5.2.4 *Let $p \in [1, \infty]$ and define the convolution operator $*$ by*

$$\left\{ \begin{array}{llll} * : & L^1(\mathbb{R}^N) \times L^p(\mathbb{R}^N) & \mapsto & L^p(\mathbb{R}^N), \\ & (u, v) & \mapsto & u * v, \ \ (u*v)(x) = \int_{\mathbb{R}^N} u(x-y)v(y)dy. \end{array} \right. \tag{5.2.1}$$

The operator $*$ is well-defined and satisfies

$$u * v \quad = \quad v * u \in L^p(\mathbb{R}^N), \ \ \|u * v\|_{L^p(\mathbb{R}^N)} \leq \|u\|_{L^1(\mathbb{R}^N)} \|v\|_{L^p(\mathbb{R}^N)}, \tag{5.2.2}$$

$$\mathrm{supp}(u * v) \quad \subset \quad \overline{\mathrm{supp}(u) + \mathrm{supp}(v)}. \tag{5.2.3}$$

Theorem 5.2.5 *Let (ρ_n) be a mollifier sequence. We have*

i) If $v \in L^p(\mathbb{R}^N)$ and $p \in [1, \infty)$ then

$$\rho_n * v \in C^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), \ \ \textit{and} \ \ \lim_{n\to\infty} \rho_n * v = v \ \ \textit{in} \ \ L^p(\mathbb{R}^N). \tag{5.2.4}$$

ii) If $v \in C^k(\mathbb{R}^N)$ and $K \subset \mathbb{R}^N$ is compact then

$$\left\{ \begin{array}{l} \rho_n * v \in C^\infty(\mathbb{R}^N), \ \ D^\alpha(\rho_n * v) = \rho_n * D^\alpha v, \ \ \forall \alpha \in \mathbb{N}_0^N, \ |\alpha| \leq k, \ \textit{and} \\ \lim_{n\to\infty} \rho_n * v = v \ \ \textit{in} \ \ C^k(K). \end{array} \right. \tag{5.2.5}$$

It follows from (5.2.3) and (5.2.5) that for $\varphi \in \mathcal{D}$, $\lim_{n\to\infty} \rho_n * \varphi = \varphi$ in $\mathcal{D}$.

For the proof of the results above and more on the convolution of functions, see, for example, [5, 20, 34].

5.2.2 Distributions

As we will see, distributions generalize the notion of a function. In the context of PDEs, they are fundamental to giving a precise meaning to PDEs, which otherwise would not have one, and to solve them.

Roughly speaking distributions can be very irregular functions. However, they have derivatives of any order.

Definition 5.2.6 *Let $\Omega \subset \mathbb{R}^N$ be an open set.*

1) A distribution in Ω is a map $T : \mathcal{D}(\Omega) \mapsto \mathbb{C}$ satisfying

(i) T is linear in the following sense:

$$\forall \varphi_1, \varphi_2 \in \mathcal{D}(\Omega), \quad \forall \alpha_1, \alpha_2 \in \mathbb{C}: \quad \langle T, \alpha_1\varphi_1 + \alpha_2\varphi_2\rangle = \alpha_1\langle T, \varphi_1\rangle + \alpha_2\langle T, \varphi_2\rangle,$$

where $\langle T, \varphi\rangle$ denotes the value of T at φ,

(ii) T is continuous in the sense:

$$\forall (\varphi_n) \ \text{in} \ \mathcal{D}(\Omega), \quad \lim_{n\to\infty} \varphi_n = 0 \ \text{in} \ \mathcal{D}(\Omega) \quad \text{implies} \quad \lim_{n\to\infty} \langle T, \varphi_n\rangle = 0.$$

2) The set of distributions in Ω is denoted by $\mathcal{D}'(\Omega)$. Often, instead of $\mathcal{D}'(\mathbb{R}^N)$ we will write $\mathcal{D}'$.

3) We say $T \in \mathcal{D}'(\Omega)$ is of finite order if

$$\begin{cases} \exists m \in \mathbb{N}, \ \forall K \subset \Omega, \ K \text{ compact}, \ \exists C > 0, \ \forall \varphi \in \mathcal{D}_K(\Omega), \\ \qquad\qquad\qquad\qquad |\langle T, \varphi\rangle| \leq C\|\varphi\|_{C^m(\Omega)}. \end{cases} \tag{5.2.6}$$

The integer m is called an "order of T". Distributions of order zero are called "measures".

The interest of distributions of finite order is that they can be identified with the dual space of $C_0^m(\Omega)$.

Example 5.2.7 *Let $f \in L^1_{loc}(\Omega)$ and define*

$$T : \mathcal{D}(\Omega) \mapsto \mathbb{C}, \quad \langle T, \varphi\rangle = \int_\Omega f(x)\varphi(x)dx. \tag{5.2.7}$$

Then $T \in \mathcal{D}'(\Omega)$. Indeed, the linearity of T is clear. If $\lim_{n\to\infty} \varphi_n = 0$ in $\mathcal{D}(\Omega)$ then

$$\begin{aligned} |\langle T, \varphi_n\rangle| &= \left|\int_\Omega f(x)\varphi_n(x)dx\right| \leq \int_\Omega |f(x)||\varphi_n|dx \\ &\leq \|f\|_{L^1(K)}\|\varphi_n\|_{C^0(K)} \\ &\xrightarrow[n\to\infty]{} 0, \end{aligned}$$

where $K \subset \Omega$ is a compact such that $\operatorname{supp}(\varphi) \subset K$ for all n. The distribution T identifies the function f uniquely. This follows from the fact that if $\int_\Omega f(x)\varphi(x)dx = 0$, for all $\varphi \in \mathcal{D}(\Omega)$, then $f = 0$ a.e. (for Lebesgue measure); see Lemma 1.3.9.

Example 5.2.7 shows that Definition 5.2.6 extends the notion of $L^1_{loc}(\Omega)$ functions. Identifying f with T_f, we write $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$. The following example shows that the inclusion is strict.

Example 5.2.8 *Let $\delta_0 : \mathcal{D}(\mathbb{R}^N) \mapsto \mathbb{C}$, $\langle \delta_0, \varphi\rangle = \varphi(0)$, be the distribution called "Dirac measure". It is easy to show that δ_0 is a distribution of order zero.*

Note that for all $p \in [1,\infty]$, we have $\delta_0 \notin L^p$, i.e. there is no $\delta_0 \in L^p$ such that $\int_{\mathbb{R}^N} \delta_0(x)\varphi(x)dx = \varphi(0)$, for all $\varphi \in \mathcal{D}$. Indeed, let us prove this for $p = 2$ (for the case p arbitrary, see [5]). Assume for the moment that $\delta_0 \in L^2$. Then there exists $\varphi_n \in \mathcal{D}$, $\lim_{n\to\infty} \varphi_n = \delta_0$ in L^2; see Theorem 1.3.5. Consider $u_n = \varphi_n(1-\eta_n)$, where $\eta_n \in \mathcal{D}$ is a $\{\{0\}, B(0,1/n)\}$ cut-off function. Assuming for the moment $\lim_{n\to\infty} u_n = \delta_0$ in L^2, we get

$$0 < \|\delta_0\|^2_{L^2(\mathbb{R}^N)} = \lim_{n\to\infty} \int_{\mathbb{R}^N} \delta_0(x)\overline{u_n}(x)dx = \lim_{n\to\infty} \langle \delta_0, \overline{u_n}\rangle = \overline{u_n}(0) = 0,$$

which is a contradiction and proves $\delta_0 \notin L^2(\mathbb{R}^N)$.

Now we prove $\lim_{n\to\infty} u_n = \delta_0$ in $L^2(\mathbb{R}^N)$. We have

$$\begin{aligned}\lim_{n\to\infty} \|\varphi_n - u_n\|^2_{L^2(\mathbb{R}^N)} = & \lim_{n\to\infty} \int_{B(0,1/n)} |\varphi_n\eta_n|^2 \le \lim_{n\to\infty} \int_{B(0,1/n)} |\varphi_n|^2 \\ \le & \; C\left(\lim_{n\to\infty} \int_{B(0,1/n)} |\varphi_n - \delta_0|^2 + \lim_{n\to\infty} \int_{B(0,1/n)} |\delta_0|^2\right) = 0,\end{aligned}$$

where we have used the Lebesgue dominated convergence theorem. This implies $\lim_{n\to\infty} u_n = \delta_0$ in $L^2(\mathbb{R}^N)$ because $\lim_{n\to\infty} \varphi_n = \delta_0$ in L^2.

$\mathcal{D}'(\Omega)$ is a vector space, and has a topology, which for the needs of this introduction will be characterized by the convergence of sequences as given by the following definition.

Definition 5.2.9 *Let T, T_1, T_2 in $\mathcal{D}'(\Omega)$, α_1, $\alpha_2 \in \mathbb{C}$, and $k \in C^\infty(\Omega)$. We define*

1) $\alpha_1 T_1 + \alpha_2 T_2 \in \mathcal{D}'(\Omega)$ by

$$\langle \alpha_1 T_1 + \alpha_2 T_2, \varphi\rangle = \alpha_1\langle T_1, \varphi\rangle + \alpha_2\langle T_2, \varphi\rangle.$$

2) $kT \in \mathcal{D}'(\Omega)$ by

$$\langle kT, \varphi\rangle = \langle T, k\varphi\rangle.$$

3) Given T and a sequence (T_n) of distributions in $\mathcal{D}'(\Omega)$, we say "T_n converges to T in $\mathcal{D}'(\Omega)$" if $\lim_{n\to\infty}\langle T_n, \varphi\rangle = \langle T, \varphi\rangle$, for all $\varphi \in \mathcal{D}(\Omega)$.

It is easy to check that 1), 2) are consistent, while the consistency[2] of 3) follows from Theorem 9.4.5 in Annex.

[2]Actually, we have the following stronger result: if $\lim_{n\to\infty}\langle T_n, \varphi\rangle$ exists for all $\varphi \in \mathcal{D}(\Omega)$ then T defined by $\langle T, \varphi\rangle := \lim_{n\to\infty}\langle T_n, \varphi\rangle$ defines a distribution in $\mathcal{D}'(\Omega)$ and $\lim_{n\to\infty} T_n = T$ in $\mathcal{D}'(\Omega)$.

Example 5.2.10 *Let* (ρ_n) *be a mollifier sequence; see Example 1.1.13. Then* $\lim\limits_{n\to\infty} \rho_n = \delta_0$ *in* $\mathcal{D}'$ *because for* $\varphi \in \mathcal{D}$*, by using* (5.2.7) *we get*

$$\langle \rho_n, \varphi\rangle = \int_{B\left(0,\frac{1}{n}\right)} \rho_n(x)\varphi(x)dx = \mathrm{Re}(\varphi(\theta_n^r)) + i\mathrm{Im}(\varphi(\theta_n^i)) = \varphi(\theta_n) \xrightarrow[n\to\infty]{} \varphi(0),$$

where θ_n^r, $\theta_n^i \in B\left(0, \frac{1}{n}\right)$.

Example 5.2.11 *Let* $a \in \mathbb{R}$ *and* $T_n = n\mathbb{1}_{\left(a,a+\frac{1}{n}\right)}$*. Then* $\lim\limits_{n\to\infty} T_n = \delta_a$ *in* $\mathcal{D}'(\mathbb{R})$ *because for* $\varphi \in \mathcal{D}(\mathbb{R})$ *we have*

$$\langle T_n, \varphi\rangle = \int_a^{a+\frac{1}{n}} n\varphi(x)dx \xrightarrow[n\to\infty]{} \varphi(a) =: \langle \delta_a, \varphi\rangle.$$

Example 5.2.12 *The convergence in* $\mathcal{D}'(\Omega)$ *generalizes (or is consistent with) the convergence in* $L^1(\Omega)$*. Indeed, let* (f_n) *in* $L^1(\Omega)$, $f \in L^1(\Omega)$, $\lim\limits_{n\to\infty} f_n = f$ *in* $L^1(\Omega)$*. Then, if* $T_n = f_n$ *we have* $\lim\limits_{n\to\infty} T_n = T$ *in* $\mathcal{D}'(\Omega)$ *because for* $\varphi \in \mathcal{D}(\Omega)$ *we have*

$$\begin{aligned} |\langle T_n, \varphi\rangle - \langle T, \varphi\rangle| &= \left|\int_\Omega (f_n(x) - f(x))\varphi(x)dx\right| \le \|f_n - f\|_{L^1(\Omega)}\|\varphi\|_{L^\infty(\Omega)} \\ &\xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Note that in general the distributions do not have meaning at any point as they are defined only in $\mathcal{D}(\Omega)$. However, it is useful to speak about the support of a distribution.

Definition 5.2.13 *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and* T, $S \in \mathcal{D}'(\Omega)$.

1) We say "$T = 0$ *in* G*", with* $G \subset \Omega$, G *open, if* $\langle T, \varphi\rangle = 0$ *for all* $\varphi \in \mathcal{D}(\Omega)$ *with* $\mathrm{supp}(\varphi) \subset G$.

2) We set $N(T)$ *to be the largest open set in* Ω *where* $T = 0$*, and call it "null set of* T*" (or "zero set of* T*").*

The support of $T \in \mathcal{D}'(\Omega)$*, denoted by* $\mathrm{supp}(T)$*, is defined by* $\mathrm{supp}(T) = \Omega\backslash N(T)$.

3) We say "$T = S$ *in* $\mathcal{D}'(\Omega)$*" if* $N(T - S) = \Omega$*, or equivalently* $\mathrm{supp}(T - S) = \emptyset$*, or even* $\langle T, \varphi\rangle = \langle S, \varphi\rangle$*, for all* $\varphi \in \mathcal{D}(\Omega)$.

Example 5.2.14 *We have* $N(\delta_0) = \mathbb{R}\backslash\{0\}$ *and* $\mathrm{supp}(\delta_0) = \{0\}$.

The following theorem shows that a distribution with compact support is of finite order, i.e. it belongs to the dual space of $C_0^m(\Omega)$ (see 3) in Definition (5.2.6), for a certain $m \in \mathbb{N}_0$.

Theorem 5.2.15 *Let $\Omega \subset \mathbb{R}^N$ be open and $T \in \mathcal{D}'(\Omega)$ be with compact support. Then T is of finite order, i.e. T satisfies (5.2.6).*

Proof. See Theorem 9.4.6 in Annex. □

This theorem shows that any $T \in \mathcal{D}'(\Omega)$ restricted in $\mathcal{D}_K(\Omega)$, $K \subset \Omega$ compact, is of finite order.

5.2.3 Derivatives of distributions

In general, a distribution is not a function. At the same time, a distribution is very regular as it has derivatives of any order as defined below.

Definition 5.2.16 *Let $T \in \mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{R}^N$ be open. The derivative of T with respect to x_i, $i = 1, \dots, N$, or the x_i-partial derivative of T, denoted by $\partial_{x_i} T$, is defined by*

$$\langle \partial_{x_i} T, \varphi \rangle = -\langle T, \partial_{x_i} \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{5.2.8}$$

If $\varphi = \varphi(x_1, x_2, \dots, x_N)$, resp. $\varphi = \varphi(x, y, \dots, z)$, sometimes instead of $\partial_{x_1} T$, $\partial_{x_2} T$, ... we will write $\partial_1 T$, $\partial_2 T$, ..., resp. $\partial_x T$, $\partial_y T$, ...

In general, if $\alpha \in \mathbb{N}_0^N$, D^α derivative of T, denoted by $D^\alpha T$, is defined by

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{5.2.9}$$

We note that Definition 5.2.16 is motivated by the usual derivative of smooth functions. For example, if $T = f \in C^1(\Omega)$ then

$$\langle \partial_i T, \varphi \rangle = \int_\Omega \partial_i f(x) \varphi(x) dx = -\int_\Omega f(x) \partial_i \varphi(x) dx = -\langle T, \partial_i \varphi \rangle,$$

where we have integrated it by parts in Ω. Hence $\partial_i T = \partial_i f$.

Also, $D^\alpha T$ of (5.2.9) is well-defined in $\mathcal{D}'(\Omega)$. Indeed, $D^\alpha T$ is linear. It is also continuous because if (φ_n) is a sequence in $\mathcal{D}(\Omega)$ converging to 0 in $\mathcal{D}(\Omega)$, then $(D^\alpha \varphi_n)$ converges also to 0 in $\mathcal{D}(\Omega)$, and therefore

$$\langle D^\alpha T, \varphi_n \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi_n \rangle \xrightarrow[n\to\infty]{} 0.$$

Example 5.2.17 *Let $\alpha \in \mathbb{N}_0^N$ and for $\varphi \in \mathcal{D}$ set $\langle T, \varphi \rangle = (-1)^{|\alpha|} D^\alpha \varphi(0)$. Then $T = D^\alpha \delta_0$, $\operatorname{supp}(T) = \{0\}$, and T is a distribution of order $|\alpha|$.*

Example 5.2.18 *Let $T = H(x) = \mathbb{1}_{(0,\infty)}(x)$, $x \in \mathbb{R}$, be the Heaviside function. Then*

$$\begin{aligned} \langle T', \varphi \rangle &= -\langle T, \varphi' \rangle = -\int_{\mathbb{R}} H(x) \varphi'(x) dx = -\int_0^\infty \varphi'(x) dx = \varphi(0) \\ &= \langle \delta_0, \varphi \rangle. \end{aligned}$$

So, $H' = \delta_0$ (here $'$ denotes the derivative w.r.t. x).

Example 5.2.19 *Let* $T = \ln|x| = \ln r,\ x \in \mathbb{R}^2,\ r = |x|$. *Then*

$$\begin{aligned}
\langle \Delta T, \varphi\rangle &:= \langle T_{xx} + T_{yy}, \varphi\rangle = \langle T, \Delta\varphi\rangle \\
&= \int_{\mathbb{R}^N} \ln|x|\Delta\varphi(x)dx \\
&= \lim_{\epsilon\to 0^+} \int_{\{|x|>\epsilon\}} \ln|x|\Delta\varphi(x)dx \\
&= \lim_{\epsilon\to 0^+} \left(\int_{\{|x|=\epsilon\}} (\varphi\partial_r \ln|x| - \ln|x|\partial_r\varphi)d\sigma + \int_{\{|x|>\epsilon\}} \varphi(x)\Delta\ln|x|dx\right) \\
&= \lim_{\epsilon\to 0^+} \int_{\{|x|=\epsilon\}} \left(\frac{1}{\epsilon}\varphi - \ln\epsilon\partial_r\varphi\right) d\sigma \\
&= 2\pi\varphi(0); \qquad \textit{hence} \\
-\Delta\left(-\frac{1}{2\pi}\ln|x|\right) &= \delta_0 \quad \textit{in } \mathscr{D}'(\mathbb{R}^2).
\end{aligned} \tag{5.2.10}$$

Example 5.2.20 *Let* $T(x) = \ln|x|,\ x \in \mathbb{R}$. *As* $\int_1^n \ln r dr = 1 + n(\ln n - 1)$ *it follows* $T \in L^1_{loc}(\mathbb{R})$ *and so* $T \in \mathscr{D}'(\mathbb{R})$. *Clearly* $T' \in \mathscr{D}'(\mathbb{R})$ *exists. However,* $(\ln|x|)' = \dfrac{1}{x}$ *and* $\dfrac{1}{x} \notin \mathscr{D}'(\mathbb{R})$. *So, necessarily* $T' \neq (\ln|x|)'$. *Let us compute* T'. *For* $\varphi \in \mathscr{D}(\mathbb{R})$, *we have*

$$\begin{aligned}
\langle T', \varphi\rangle &= -\langle T, \varphi'\rangle = -\int_{\mathbb{R}} \ln|x|\varphi'(x)dx = -\lim_{\epsilon\to 0^+}\int_{\{|x|>\epsilon\}} \ln|x|\varphi'(x)dx \\
&= \lim_{\epsilon\to 0^+}\left((\varphi(\epsilon) - \varphi(-\epsilon))\ln\epsilon + \int_{\{|x|>\epsilon\}} \frac{\varphi(x)}{x}dx\right) \\
&= \lim_{\epsilon\to 0^+}\int_{\{|x|>\epsilon\}} \frac{\varphi(x)}{x}dx.
\end{aligned}$$

We define $\mathrm{pv}(x^{-1}) \in \mathscr{D}'(\mathbb{R})$ *by*

$$\langle \mathrm{pv}(x^{-1}), \varphi\rangle = \lim_{\epsilon\to 0^+}\int_{\{|x|>\epsilon\}} \frac{\varphi(x)}{x}dx. \tag{5.2.11}$$

So we have proved

$$(\ln|x|)' = \langle \mathrm{pv}(x^{-1}), \varphi\rangle \ \textit{ in } \mathscr{D}'(\mathbb{R}). \tag{5.2.12}$$

The definition (5.2.11) *defines* $\mathrm{pv}(x^{-1}) \in \mathscr{D}'(\mathbb{R})$ *because for* $\varphi \in \mathscr{D}(\mathbb{R})$ *we have*

$$\begin{aligned}
\langle \mathrm{pv}(x^{-1}), \varphi\rangle &= \lim_{\epsilon\to 0^+}\int_{\{\epsilon<|x|<M\}} \frac{\varphi(x)-\varphi(0)}{x}dx = \int_{\{|x|<M\}} \frac{\varphi(x)-\varphi(0)}{x}dx, \quad \textit{so} \\
|\langle \mathrm{pv}(x^{-1}), \varphi\rangle| &\leq (2M)\|\varphi\|_{C^1(\mathbb{R})},
\end{aligned}$$

where $M = M(\varphi) > 0$ is such that $\operatorname{supp}(\varphi) \subset [-M, M]$*. It follows furthermore that* $\operatorname{pv}(x^{-1})$ *is of order one (so,* $\operatorname{pv}(x^{-1})$ *is not a measure).*

The distribution $\operatorname{pv}(x^{-1})$ *is called a "principal value of x^{-1}". Its name is motivated by the fact that it corresponds to the Cauchy principal value of* $\displaystyle\int_{\mathbb{R}} \frac{\varphi(x)}{x} dx$*. Finally, note that* $\operatorname{pv}(x^{-1})$ *is equivalently defined by*

$$\langle \operatorname{pv}(x^{-1}), \varphi \rangle = \int_{\{|x|>\epsilon\}} \frac{\varphi(x)}{x} dx + \int_{\{|x|<\epsilon\}} \frac{\varphi(x) - \varphi(0)}{x} dx, \quad \forall \epsilon > 0. \tag{5.2.13}$$

Example 5.2.21 *Let $\Omega \subset \mathbb{R}^N$ be open, simply connected, and look for $T \in \mathcal{D}'(\Omega)$:*

$$\nabla T := (\partial_i T) = 0, \quad i.e. \quad \langle T, \partial_i \varphi \rangle = 0 \ \text{ for every } \ \varphi \in \mathcal{D}(\Omega). \tag{5.2.14}$$

We show that T is equal to a constant. For simplicity and without loss of generality, we assume $\Omega = (a_1, b_1) \times \cdots \times (a_N, b_N) =: I_1 \times \cdots \times I_N$.

We note that a function $\phi \in \mathcal{D}(\Omega)$ is the ∂_i derivative of a certain function $\varphi \in \mathcal{D}(\Omega)$ if and only if

$$\int_{a_i}^{b_i} \phi(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N) dx_i = 0.$$

One direction of this statement is clear and the other follows by considering

$$\varphi(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N) = \int_{a_i}^{x_i} \phi(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N) dt.$$

Next we choose $\eta_i \in \mathcal{D}(I_i)$, with $\int_{I_i} \eta_i(x_i) dx_i = 1$, for all $i = 1, \ldots, N$. Then for every $\varphi =: \varphi_1 \in \mathcal{D}(\Omega)$ and $x = (x_1, \ldots, x_N) \in \Omega$ define φ_2 and ϕ_1 by

$$\begin{aligned} \varphi_2(x_2, \ldots, x_N) &:= \int_{a_1}^{b_1} \varphi_1(x_1, x_2, \ldots, x_N) dx_1, \\ \varphi_1(x_1, \ldots, x_N) &= \phi_1(x_1, \ldots, x_N) + \eta_1(x_1) \varphi_2(x_2, \ldots, x_N). \end{aligned}$$

It is easy to show that $\eta_1 \varphi_2 \in \mathcal{D}(\Omega)$, $\phi_1 \in \mathcal{D}(\Omega)$, and $\int_{a_1}^{b_1} \phi_1(x_1, x_2, \ldots, x_N) dx_1 = 0$. So ϕ_1 is equal to the ∂_1 derivative of a test function in Ω. Therefore $\langle T, \phi_1 \rangle = 0$ and

$$\langle T, \varphi_1 \rangle = \langle T, \eta_1 \varphi_2 \rangle.$$

We proceed with φ_2 as with φ_1. So we consider φ_3 and ϕ_2 defined by

$$\begin{aligned} \varphi_3(x_3, \ldots, x_N) &= \int_{a_2}^{b_2} \varphi_2(x_2, x_3, \ldots, x_N) dx_2, \\ \varphi_2(x_2, \ldots, x_N) &= \phi_2(x_2, \ldots, x_N) + \eta_2(x_2) \varphi_3(x_3, \ldots, x_N). \end{aligned}$$

Therefore we have

$$\eta_1 \varphi_2 = \eta_1 \phi_2 + \eta_1 \eta_2 \varphi_3.$$

Clearly $\eta_1\eta_2\varphi_3 \in \mathcal{D}(\Omega)$, $\eta_1\phi_2 \in \mathcal{D}(\Omega)$, *and* $\int_{a_2}^{b_2}\eta_1(x_1)\phi_2(x_2, x_3, \ldots, x_N)dx_2 = 0$. *Therefore* $\eta_1\phi_2$ *is equal to the* ∂_2 *derivative of a test function in* Ω. *So* $\langle T, \eta_1\phi_2\rangle = 0$ *and*

$$\langle T, \eta_1\varphi_2\rangle \quad = \quad \langle T, \eta_1\eta_2\varphi_3\rangle.$$

Continuing in this way we get

$$\begin{aligned}
\langle T, \varphi\rangle &= \langle T, \varphi_1\rangle \\
&= \langle T, \eta_1\varphi_2\rangle \\
&\vdots \\
&= \langle T, \eta_1\eta_2\cdots\eta_N\varphi_{N+1}\rangle \\
&= \left\langle T, \eta_1(x_1)\cdots\eta_N(x_N)\textstyle\int_{a_N}^{b_N}\varphi_N(x_N)dx_N\right\rangle \\
&= \left\langle T, \eta_1(x_1)\cdots\eta_N(x_N)\textstyle\int_{a_N}^{b_N}\int_{a_{N-1}}^{b_{N-1}}\varphi_{N-1}(x_{N-1}, t_N)dt_{N-1}dx_N\right\rangle \\
&\vdots \\
&= \left\langle T, \eta_1(x_1)\cdots\eta_N(x_N)\textstyle\int_{a_N}^{b_N}\int_{a_{N-1}}^{b_{N-1}}\cdots\int_{a_1}^{b_1}\varphi_1(x_1, \ldots, x_{N-1}, x_N)dx_1\cdots dx_{N-1}dx_N\right\rangle \\
&= \langle T, \eta_1(x_1)\cdots\eta_N(x_N)\langle 1, \varphi\rangle\rangle \\
&= \langle T, \eta_1(x_1)\cdots\eta_N(x_N)\rangle\,\langle 1, \varphi\rangle \\
&= \langle\langle T, \eta_1(x_1)\cdots\eta_N(x_N)\rangle, \varphi\rangle\,,
\end{aligned}$$

which proves $T = \langle T, \eta_1(x_1)\cdots\eta_N(x_N)\rangle \in \mathbb{C}$.

5.3 Convolution of distributions and fundamental solutions

All throughout this section we consider the space of test functions, resp. distributions, in $\mathbb{R}^N$, which is denoted by $\mathcal{D}$, resp. $\mathcal{D}'$. We have seen the convolution operator $*$ is $L^2(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$; see (5.2.1). In this section we will extend it in $\mathcal{D}' \times \mathcal{D}'$ and use it to solve certain PDEs. For more details, we refer the reader to [18, 22, 23].

Definition 5.3.1 *Let* $T, S \in \mathcal{D}'$ *with at least one of them, for example* S, *with compact support. Then we define* $T * S \in \mathcal{D}'$, *called "convolution of* T *with* S*", by*

$$\langle T * S, \varphi\rangle = \langle T(x), \langle S(y), \varphi(x+y)\rangle\rangle, \quad \forall\varphi \in \mathcal{D}. \tag{5.3.1}$$

We note that the definition of $\langle T * S, \varphi\rangle$ is motivated by the convolution of functions. For example, if $T = f$ and $S = g$ with $f, g \in L^1$, then (see (5.2.1))

$$\langle T * S, \varphi\rangle \quad = \quad \int_{\mathbb{R}^N}(T * S)(z)\varphi(z)dz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(z-y)g(y)\varphi(z)dydz$$

$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)g(y)\varphi(x+y)dydx = \langle f(x), \langle g(y), \varphi(x+y)\rangle\rangle.$$

Also, the definition (5.3.1) is consistent, i.e. $T * S \in \mathscr{D}'$; see Proposition 9.4.10 in Annex.

Example 5.3.2 *Let $T \in \mathscr{D}'$. Then $T * \delta_0 = T$ because for every $\varphi \in \mathscr{D}$ we have*

$$\langle T * \delta_0, \varphi\rangle = \langle T(x), \langle \delta_0(y), \varphi(x+y)\rangle\rangle = \langle T(x), \varphi(x)\rangle.$$

The following theorem shows the density of $\mathscr{D}$ in $\mathscr{D}'$ and that the convolution is commutative. The proof uses the lemmas 9.4.8 and 9.4.9 in section 9.4.4, which show that the differentiation and the integration commute with distribution pairing.

Theorem 5.3.3 *For every $T \in \mathscr{D}'$ there exists (T_n) in $\mathscr{D}$ converging to T in $\mathscr{D}'$, i.e.*

$$\mathscr{D} \subset \mathscr{D}' \quad \textit{densely.} \tag{5.3.2}$$

Furthermore, if $T, S \in \mathscr{D}'$, with for example S with compact support, then

$$T * S = S * T, \tag{5.3.3}$$

$$D^\alpha(T * S) = D^\alpha T * S = T * D^\alpha S, \quad \forall \alpha \in \mathbb{N}_0^N. \tag{5.3.4}$$

Proof. For (5.3.2), consider $\eta \in \mathscr{D}$ a $\{\overline{B}(0,1), B(0,2)\}$ cut-off function, and set $\eta_n(x) = \eta(x/n)$. It is clear that $\eta_n T$ has compact support and $\lim_{n\to\infty}(\eta_n T) = T$ in $\mathscr{D}'$. Therefore, without loss of generality we may assume T has compact support. Set $T_n = T * \rho_n$, where (ρ_n) is a mollifier sequence. Then

$$\begin{aligned}
\langle T_n, \varphi\rangle &= \langle T * \rho_n, \varphi\rangle = \langle T(x), \langle \rho_n(y), \varphi(x+y)\rangle\rangle \\
&= \langle T(x), \check{\rho}_n * \varphi\rangle \\
&\xrightarrow[n\to\infty]{} \langle T, \varphi\rangle,
\end{aligned}$$

where we used the fact that if (ρ_n) is a mollifier sequence then $(\check{\rho}_n)$, with $\check{\rho}_n(x) = \rho_n(-x)$, is also a mollifier sequence and $\lim_{n\to\infty} \check{\rho}_n * \varphi = \varphi$ in $\mathscr{D}$ (see Theorem 5.2.5).

For the equality (5.3.3) we note that from Remark 9.4.7 we have $S = \eta S$, with a certain $\eta \in \mathscr{D}$. Then considering $T_n = T * \rho_n$, using Lemma 9.4.8 and Lemma 9.4.9 implies

$$\begin{aligned}
\langle T * S, \varphi\rangle &= \langle T(x), \langle S(y), \varphi(x+y)\rangle\rangle \\
&= \langle T(x), \langle S(y), \eta(y)\varphi(x+y)\rangle\rangle \\
&= \lim_{n\to\infty} \langle T_n(x), \langle S(y), \eta(y)\varphi(x+y)\rangle\rangle \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} T_n(x)\langle S(y), \eta(y)\varphi(x+y)\rangle dx \\
&= \lim_{n\to\infty} \left\langle S(y), \int_{\mathbb{R}^N} T_n(x)\eta(y)\varphi(x+y)dx \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \langle S(y), \langle (T * \rho_n)(x), \eta(y)\varphi(y+x)\rangle\rangle \\
&= \lim_{n\to\infty} \langle S(y), \langle T(x), \langle \rho_n(z), \eta(y)\varphi(y+x+z)\rangle\rangle\rangle.
\end{aligned}
$$

Now we use the fact $\lim_{n\to\infty}\langle \rho_n(z), \varphi(y+x+z)\rangle = \varphi(y+x)$ in $\mathscr{D}$, see the comment after Theorem 5.2.4, which by using Lemma 9.4.8 implies $\lim_{n\to\infty}\langle T(x), \langle \rho_n(z), \eta(y)\varphi(y+x+z)\rangle\rangle = \langle T(x), \eta(y)\varphi(y+x)\rangle$ in $\mathscr{D}$, to conclude that

$$
\begin{aligned}
\langle T * S, \varphi\rangle &= \langle S(y), \langle T(x), \eta(y)\varphi(y+x)\rangle\rangle \\
&= \langle \eta(y)S(y), \langle T(x), \varphi(y+x)\rangle\rangle \\
&= \langle S * T, \varphi\rangle.
\end{aligned}
$$

Finally (5.3.4) follows from Lemma 9.4.8 as follows:

$$
\begin{aligned}
D^\alpha\langle T * S, \varphi\rangle &= (-1)^{|\alpha|}\langle T(x), D_x^\alpha\langle S(y), \varphi(x+y)\rangle\rangle = (-1)^{|\alpha|}\langle T(x), \langle S(y), D_x^\alpha\varphi(x+y)\rangle\rangle \\
&= (-1)^{|\alpha|}\langle T(x), \langle S(y), D_y^\alpha\varphi(x+y)\rangle\rangle = \langle T(x), \langle D_y^\alpha S(y), \varphi(x+y)\rangle\rangle \\
&= \langle T * D^\alpha S, \varphi\rangle,
\end{aligned}
$$

which completes the proof. □

By using the convolution and the so-called "*fundamental solutions*", one can solve linear PDEs with constant coefficients as follows.

Definition 5.3.4 *Let L be a m-th order linear PDE operator with constant coefficients,*

$$
L = \sum_{|\alpha|\le m} c_\alpha D^\alpha, \quad c_\alpha \in \mathbb{C}.
$$

A fundamental solution to L is a distribution $E \in \mathscr{D}'$ such that

$$
LE := \sum_{|\alpha|\le m} c_\alpha D^\alpha E = \delta_0. \tag{5.3.5}
$$

As a corollary of Theorem 5.3.3, we have the following result.

Theorem 5.3.5 *Let L be a m-th order linear PDE operator with constant coefficients, and let $E \in \mathscr{D}'$ be a fundamental solution to L. For $f \in \mathscr{D}'$, if at least one of E and f has compact support, then*

$$
L(f * E) = f * LE = f. \tag{5.3.6}
$$

So $u = f * E$ solves $Lu = f$ in $\mathscr{D}'$.

Proof. The proof follows directly from Theorem 5.3.3 and Example 5.3.2:

$$
L(f * E) = f * LE = f * \delta_0 = f.
$$

□

Example 5.3.6 *We have seen that if $u \in C^2(\Omega)$ with $\Delta u = 0$ in Ω then $u \in C^\infty(\Omega)$. We have a stronger result: if $u \in L^1(\mathbb{R}^N)$ and $\Delta u = 0$ in $\mathscr{D}'(\mathbb{R}^N)$ then $u \in C^\infty(\mathbb{R}^N)$.*

Indeed, let $u_n = \rho_n * u$, where (ρ_n) is a mollifier sequence. Then from theorems 5.2.4 and 5.2.5 we have (u_n) is uniformly bounded in $L^1(\mathbb{R}^N)$, $u_n \in C^\infty(\mathbb{R}^N)$, $\lim_{n\to\infty} u_n = u$ in $L^1(\mathbb{R}^N)$, and $\Delta u_n = \rho_n * \Delta u = 0$. Also, from Problem 4.13 it follows that (u_n) is uniformly bounded. Then (u_n) satisfies the conditions of Theorem 4.4.6 in every ball in $\mathbb{R}^N$, which implies that $u \in C^\infty(\mathbb{R}^N)$.

As another application of the convolution and the fundamental solution to a PDE, we will present two examples involving heat and wave equations in dimension one. Here, calculus are without rigor, and for a rigorous proof we refer the reader to any specialized monograph on distributions, for example [3, 22].

Example 5.3.7 *Let $G(\cdot,t) \in C^0([0,\infty);\mathscr{D}'(\mathbb{R})) \cap C^1((0,\infty);\mathscr{D}'(\mathbb{R}))$ (see footnote[3]) satisfying*

$$\partial_t G(\cdot,t) - \Delta G(\cdot,t) = 0 \quad \text{for } t>0, \quad G(\cdot,0) = \delta_0(\cdot) \ \text{ in } \mathscr{D}'(\mathbb{R}). \tag{5.3.7}$$

The distribution G is called a "fundamental solution to the heat equation". Let u be given by

$$u(\cdot,t) = G(\cdot,t) * u_0(\cdot) + \int_0^t G(\cdot,t-s) * f(\cdot,s)ds, \quad \text{in } \mathscr{D}'(\mathbb{R}). \tag{5.3.8}$$

Then $u(\cdot,t) \in C^1((0,\infty);\mathscr{D}'(\mathbb{R})) \cap C^0([0,\infty);\mathscr{D}'(\mathbb{R}))$ (see footnote[3]), provided $u_0 \in \mathscr{D}'(\mathbb{R})$ and $f(\cdot,t) \in C^0((0,\infty);\mathscr{D}'(\mathbb{R}))$ are with compact support. Furthermore, u solves[4]

$$\partial_t u(\cdot,t) - \Delta u(\cdot,t) = f(\cdot,t) \quad \text{for } t>0, \quad u(\cdot,0) = u_0(\cdot) \ \text{ in } \mathscr{D}'(\mathbb{R}). \tag{5.3.9}$$

Indeed, by direct computations we obtain

$$\begin{aligned}
\partial_t u(\cdot,t) - \Delta u(\cdot,t) &= \partial_t G(\cdot,t) * u_0 + G(\cdot,0) * f(\cdot,t) + \int_0^t \partial_t G(\cdot,t-s) * f(\cdot,s)ds \\
&- \Delta G(\cdot,t) * u_0 - \int_0^t \Delta G(\cdot,t-s) * f(\cdot,s)ds \\
&= (\partial_t G(\cdot,t) - \Delta G(\cdot,t)) * u_0 \\
&+ \delta_0 * f(\cdot,t) \\
&+ \int_0^t (\partial_t G(\cdot,t-s) - \Delta G(\cdot,t-s)) * f(\cdot,s)ds \\
&= f(\cdot,t),
\end{aligned}$$

[3]This means (i) $t \in [0,\infty) \mapsto G(\cdot,t) \in \mathscr{D}'(\mathbb{R})$ is continuous, which is $\lim_{h\to0} G(\cdot,t+h) = G(\cdot,t)$ in $\mathscr{D}'(\mathbb{R})$, and (ii) $\partial_t G(\cdot,t) = \lim_{h\to0}(G(\cdot,t+h) - G(\cdot,t))/h$ exists in $\mathscr{D}'(\mathbb{R})$ and $t \in (0,\infty) \mapsto \partial_t G(\cdot,t) \in \mathscr{D}'(\mathbb{R})$ is continuous at every $t>0$.

[4]The formula for u in (5.3.8) holds also in $\mathscr{D}'(\mathbb{R}^N)$ and u solves (5.3.9) in $\mathscr{D}'(\mathbb{R}^N)$.

$$u(\cdot,0) \;=\; u_0(\cdot).$$

One can solve G in (5.3.7) and find (see Example 5.4.18)

$$G(x,t) \;=\; \frac{1}{\sqrt{4\pi t}}e^{-\frac{1}{4}\frac{x^2}{t}}. \tag{5.3.10}$$

Example 5.3.8 *Let $G = G(\cdot,t) \in C^2((0,\infty);\mathcal{D}'(\mathbb{R})) \cap C^1([0,\infty);\mathcal{D}'(\mathbb{R}))$ the solution to*

$$\partial_{tt}G(\cdot,t) - \Delta G(\cdot,t) = 0 \;\; \text{for} \;\; t>0, \;\; G(\cdot,0)=0, \;\; \partial_t G(\cdot,0) = \delta_0(\cdot), \;\; \text{in } \mathcal{D}'(\mathbb{R}). \tag{5.3.11}$$

G be called a "fundamental solution to the wave equation". Now we define

$$u(\cdot,t) = \partial_t G(\cdot,t) * u_0(\cdot) + G(\cdot,t) * u_1(\cdot) + \int_0^t G(\cdot,t-s) * f(\cdot,s)ds, \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{5.3.12}$$

If $u_0, u_1 \in \mathcal{D}'(\mathbb{R})$ and $f(\cdot,t) \in C^0((0,\infty);\mathcal{D}'(\mathbb{R}))$ have compact support then $u(\cdot,t) \in C^2((0,\infty);\mathcal{D}'(\mathbb{R})) \cap C^1([0,\infty);\mathcal{D}'(\mathbb{R}))$, and solve

$$\partial_{tt}u(\cdot,t) - \Delta u(\cdot,t) = f(\cdot,t) \;\; \text{for} \;\; t>0, \quad u(\cdot,0)=u_0, \quad \partial_t u(\cdot,0) = u_1. \tag{5.3.13}$$

Indeed by direct computations we obtain

$$\begin{aligned}
u_{tt}(\cdot,t) - \Delta u(\cdot,t) \;&=\; \partial_{ttt}G(\cdot,t) * u_0(\cdot) + \partial_{tt}G(\cdot,t) * u_1(\cdot) \\
&+\; \partial_t G(\cdot,0) * f(\cdot,t) + \int_0^t \partial_{tt}G(\cdot,t-s) * f(\cdot,s)ds \\
&-\; \Delta\partial_t G(\cdot,t) * u_0(\cdot) - \Delta G(\cdot,t) * u_1(\cdot) - \int_0^t \Delta G(\cdot,t-s) * f(\cdot,s)ds \\
&=\; \partial_t(\partial_{tt}G(\cdot,t) - \Delta G(\cdot,t)) * u_0(\cdot) + (\partial_{tt}G(\cdot,t) - \Delta G(\cdot,t)) * u_1(\cdot) \\
&+\; \partial_t G(\cdot,0) * f(\cdot,t) + \int_0^t (\partial_{tt}G(\cdot,t-s) - \Delta G(\cdot,t-s)) * f(\cdot,s)ds \\
&=\; f(\cdot,t), \\
u(\cdot,0) \;&=\; u_0, \quad \partial_t u(\cdot,0) = u_1.
\end{aligned}$$

The solution to (5.3.11) is given by (see Example 5.4.18)

$$\begin{aligned}
G(x,t) \;&=\; \int \frac{1}{2}(\delta_0(x+t) + \delta_0(x-t))dt = \frac{1}{2}(H(x+t) - H(x-t)) \\
&=\; \begin{cases} \frac{1}{2}, & \text{for } |x|<t, \\ 0, & \text{for } |x|>t. \end{cases}
\end{aligned} \tag{5.3.14}$$

5.4 Tempered distributions and Fourier transform

The set $\mathcal{S}'$ of tempered distributions consists of linear continuous maps in $\mathcal{S}$, where $\mathcal{S}$ is the set of infinitely differentiable functions with rapid decay (at infinity). As such, a tempered distribution generalizes the concept of distribution. Fourier transform $\mathcal{F}$, which is initially defined in $\mathcal{S}$, is invertible in $\mathcal{S}$ and $\mathcal{S}'$, and transforms a PDE with constant coefficient to a polynomial. These properties make $\mathcal{F}$ a powerful tool for the study of PDEs.

Definition 5.4.1 *Let C^∞ denote the set of continuous infinitely differentiable functions in $\mathbb{R}^N$ with complex values.*

(i) Define $\mathcal{S}$, sometimes denoted by $\mathcal{S}(\mathbb{R}^N)$ to avoid any ambiguity, by

$$\mathcal{S} = \left\{ \varphi \in C^\infty, \ \forall m \in \mathbb{N}, \ \ s_m(\varphi) := \sum_{|\alpha|\le m,\, |\beta|\le m} \|x^\alpha D^\beta \varphi\|_{C^0(\mathbb{R}^N)} < \infty \right\}.$$

The set $\mathcal{S}$ is called a "space of test functions with fast decay (at ∞)".

(ii) If $\varphi_n, \varphi \in \mathcal{S}$, $n \in \mathbb{N}$, we say "φ_n converges to φ in $\mathcal{S}$", and write "$\lim_{n\to\infty} \varphi_n = \varphi$ in $\mathcal{S}$", if for every $m \in \mathbb{N}$ we have

$$\lim_{n\to\infty} s_m(\varphi_n - \varphi) = 0.$$

Note that $\mathcal{S}$ is a vector space because $\alpha_1\varphi_1 + \alpha_2\varphi_2 \in \mathcal{S}$ for arbitrary α_1, α_2 complex constants and φ_1, $\varphi_2 \in \mathcal{S}$.

Example 5.4.2 *Let $u(x) = e^{-x^2}$ and $v(x) = e^{x^2}$. Then $u \in \mathcal{S}$ and $v \notin \mathcal{S}$.*

Definition 5.4.3 *The space of tempered distributions and the convergence in it are defined as follows.*

(i) A tempered distribution is a map $T : \mathcal{S} \mapsto \mathbb{C}$ satisfying

1) T is linear

$$\forall \varphi_1, \varphi_2 \in \mathcal{S}, \quad \forall \alpha_1, \alpha_2 \in \mathbb{C} : \quad \langle T, \alpha_1\varphi_1 + \alpha_2\varphi_2\rangle = \alpha_1\langle T, \varphi_1\rangle + \alpha_2\langle T, \varphi_2\rangle,$$

2) T is continuous in the sense

$$\forall (\varphi_n) \in \mathcal{S}, \ \lim_{n\to\infty} \varphi_n = 0 \ \text{ in } \ \mathcal{S} \quad \text{implies} \ \lim_{n\to\infty} \langle T, \varphi_n\rangle = 0.$$

The set of tempered distributions in $\mathbb{R}^N$ is denoted by $\mathcal{S}'$, or to avoid any ambiguity by $\mathcal{S}'(\mathbb{R}^N)$.

(ii) For $T, S \in \mathcal{S}'$ we say "$T = S$" if $\langle T - S, \varphi\rangle = 0$ for all $\varphi \in \mathcal{S}$.

(iii) For $T, T_n \in \mathcal{S}'$, $n \in \mathbb{N}$, we say "T_n converges to T in $\mathcal{S}'$", and write "$\lim\limits_{n\to\infty} T_n = T$ in $\mathcal{S}'$", if[5]

$$\lim_{n\to\infty} \langle T_n - T, \varphi\rangle = 0, \quad \text{for all } \varphi \in \mathcal{S}.$$

The following proposition shows that $\mathcal{S}$ is invariant with respect to the derivative of any order and multiplication by any polynomial. Furthermore, the inclusions $\mathcal{D} \subset \mathcal{S} \subset L^p \subset \mathcal{S}' \subset \mathcal{D}'$, $p \in [1, \infty)$, hold and are dense.[6]

Proposition 5.4.4 *Let α, $\beta \in \mathbb{N}_0^N$ and $p \in [1, \infty)$. Then*

$$i)\, \{x^\alpha D^\beta u,\ u \in \mathcal{S}\} \subset \mathcal{S} \text{ and,} \tag{5.4.1}$$
$$ii)\, \mathcal{D} \subset \mathcal{S} \subset L^p \subset \mathcal{S}' \subset \mathcal{D}', \tag{5.4.2}$$

with all the inclusions dense, the third inclusion being dense even for $p = \infty$.

Proof. See Proposition 9.4.11 in Annex. □

5.4.1 Fourier transform

We now introduce the Fourier transform, which we will use for solving certain PDEs and while studying Sobolev spaces.

Definition 5.4.5 *Let $u \in L^1(\mathbb{R}^N)$. Fourier transform of u, denoted by $\mathcal{F}[u]$ or $\hat{u}$, is defined by*

$$\mathcal{F} : \mathbb{R}^N \mapsto \mathbb{C}, \quad \mathcal{F}[u](\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i(\xi\cdot x)} u(x)dx, \quad \forall \xi \in \mathbb{R}^N. \tag{5.4.3}$$

Here $i^2 = -1$ and $\xi \cdot x = \sum\limits_{n=1,N} \xi_n x_n$.

The following properties in $\mathcal{S}$ are very important. In plain terms, they show that after applying $\mathcal{F}$, every operator $D^\alpha u$ becomes $\xi^\alpha \hat{u}$ in Fourier space. We will see that this property holds for tempered distributions as well and leads to the conclusion that a linear partial differential operator with constant coefficients becomes a linear equation with a polynomial coefficient in Fourier space; see Remark 5.4.12.

[5]This definition is consistent, because actually the following holds: Assume $\lim_{n\to\infty}\langle T_n, \varphi\rangle$ exists for all $\varphi \in \mathcal{S}$ and define $T : \mathcal{S} \mapsto \mathbb{C}$, $\langle T, \varphi\rangle := \lim_{n\to\infty}\langle T_n, \varphi\rangle$. Then $T \in \mathcal{S}'$ and $T_n \to T$ in $\mathcal{S}'$. See [22].

[6]If A and B are sets and B is endowed with a convergence of sequences, we say "*the inclusion $A \subset B$ is dense*" if $A \subset B$ and for every $b \in B$ there exists a sequence (a_n) in A converging to b in B.

Proposition 5.4.6 *For all $u \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{N}_0^N$ we have $\mathcal{F}[u] \in \mathcal{S}$ and*

$$\begin{aligned} D^\alpha \mathcal{F}[u](\xi) &= \mathcal{F}[(-ix)^\alpha u](\xi), &(5.4.4)\\ (i\xi)^\beta \mathcal{F}[u](\xi) &= \mathcal{F}[D^\beta u](\xi), &(5.4.5)\\ (i\xi)^\beta D^\alpha \mathcal{F}[u](\xi) &= \mathcal{F}[D^\beta((-ix)^\alpha u)](\xi). &(5.4.6)\end{aligned}$$

Proof. Let $u \in \mathcal{S}$. From (5.4.1) we have $x^\alpha D^\beta u \in L^1$ for all α, $\beta \in \mathbb{N}_0^N$. Hence, all the terms on the right-hand sides of (5.4.4), (5.4.5), and (5.4.6) are well-defined, and in particular $\mathcal{F}[u]$ is well-defined.

To prove (5.4.4), for $\alpha = (1, 0, \ldots, 0)$ for example, one can write

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h}(\mathcal{F}[u](\xi + he_1) - \mathcal{F}[u](\xi)) &= \frac{1}{(2\pi)^{N/2}} \lim_{h\to 0} \int_{\mathbb{R}^N} e^{-i(\xi\cdot x)}(e^{-ih(e_1\cdot x)} - 1)u(x)dx \\ &\xrightarrow[h\to 0]{} \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i(\xi\cdot x)}(-ix_1)u(x)dx \\ &= \mathcal{F}[(-ix)^\alpha u](\xi), \end{aligned}$$

where for the limit we used Lebesgue DCT. The proof for arbitrary α is made by recurrence. For the proof of (5.4.5) let (u_n) be a sequence in $\mathcal{D}$ converging to u in $\mathcal{S}$ and in L^1; see Proposition 5.4.4. Then by passing in limit, integrating by parts, and passing in limit again gives

$$\begin{aligned} \mathcal{F}[D^\beta u](\xi) &= \lim_{n\to\infty} \mathcal{F}[D^\beta u_n](\xi) = \frac{1}{(2\pi)^{N/2}} \lim_{n\to\infty} \int_{\mathbb{R}^N} e^{-i(\xi\cdot x)} D^\beta u_n(x)dx \\ &= \frac{(-1)^{|\beta|}}{(2\pi)^{N/2}} \lim_{n\to\infty} \int_{\mathbb{R}^N} D_x^\beta e^{-i(\xi\cdot x)} u_n(x)dx \\ &= \frac{1}{(2\pi)^{N/2}} \lim_{n\to\infty} \int_{\mathbb{R}^N} (i\xi)^\beta e^{-i(\xi\cdot x)} u_n(x)dx \\ &= (i\xi)^\beta \mathcal{F}[u](\xi). \end{aligned}$$

Finally, (5.4.6) is proved by combining (5.4.4), (5.4.5). It implies $\mathcal{F}[u] \in \mathcal{S}$ because the right-hand side of (5.4.6) is bounded

$$\left|\mathcal{F}[D^\beta((-ix)^\alpha u)](\xi)\right| \le \frac{1}{(2\pi)^{N/2}} \|D^\beta((-ix)^\alpha u)\|_{L^1(\mathbb{R}^N)} < \infty.$$

Example 5.4.7 *Given $a \in \mathbb{R}$, $a \neq 0$, we will find $\mathcal{F}[e^{-ax^2}]$, $x \in \mathbb{R}^N$. First consider $u(x) = e^{-x^2}$, $x \in \mathbb{R}^N$ and let us find $\hat{u}$. Note that $u \in \mathcal{S}$, so $\hat{u} \in \mathcal{S}$. In the case $N = 1$, by using Proposition 5.4.6 we get*

$$2xu + u' = 0, \qquad 2\hat{u}' + \xi\hat{u} = 0.$$

Hence, $\hat{u} = Ae^{-\frac{1}{4}\xi^2}$ with $A = \hat{u}(0) = \dfrac{1}{(2\pi)^{1/2}} \displaystyle\int_{\mathbb{R}^N} e^{-x^2}dx$. Note that

$$\int_{\mathbb{R}} e^{-x^2}dx = \left(\int_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy\right)^{1/2} = \left(\int_0^{2\pi}\int_0^\infty re^{-r^2}drd\theta\right)^{1/2} = \pi^{1/2}. \quad (5.4.7)$$

Therefore, $A = \dfrac{1}{2^{1/2}}$ *and*

$$\mathcal{F}[e^{-x^2}](\xi) = \frac{1}{2^{1/2}} e^{-\frac{1}{4}\xi^2}, \quad \xi \in \mathbb{R}. \tag{5.4.8}$$

In the case $N \in \mathbb{N}$*, taking into account* (5.4.8) *we have*

$$\mathcal{F}[e^{-x^2}](\xi) = \prod_{k=1,N} \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-i(\xi_k x_k)} e^{-x_k^2} dx_k = \prod_{k=1,N} \mathcal{F}[e^{-x_k^2}](\xi_k) = \prod_{k=1,N} \frac{1}{2^{1/2}} e^{-\frac{1}{4}\xi_k^2}.$$

Hence

$$\mathcal{F}[e^{-x^2}](\xi) = \frac{1}{2^{N/2}} e^{-\frac{1}{4}\xi^2}, \quad \xi \in \mathbb{R}^N. \tag{5.4.9}$$

Finally, by using (5.4.9) *and the change of variable* $y = \sqrt{a}x$*,* $a > 0$*, we get*

$$\mathcal{F}[e^{-ax^2}](\xi) = \frac{1}{(2a)^{N/2}} e^{-\frac{1}{4a}\xi^2}. \tag{5.4.10}$$

One may ask whether $\mathcal{F}[u] \in \mathcal{D}$ for any $u \in \mathcal{D}$. By using the analyticity of $\mathcal{F}[u]$, one can prove that $\mathcal{F}[u] \in \mathcal{D}$ if and only if $u = 0$. The introduction of $\mathcal{S}$ has been motivated, among others, by the idea of finding a non-trivial space of test functions which is invariant under $\mathcal{F}$. It turns out that $\mathcal{F}[\mathcal{S}] = \mathcal{S}$, as the following theorem shows.

Theorem 5.4.8 *Fourier transform* $\mathcal{F}$ *is a linear continuous*[7] *bijection from* $\mathcal{S}$ *to itself. Its inverse, denoted by* $\mathcal{F}^{-1}$*, is given by*

$$\mathcal{F}^{-1}[U](x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i(x\cdot\xi)} U(\xi) d\xi = \mathcal{F}[U](-x) = \mathcal{F}[\check{U}](x), \quad \forall U \in \mathcal{S}, \tag{5.4.11}$$

where $\check{w}(\xi) = w(-\xi)$ *for every function* w*. Furthermore, similar to* (5.4.4)*,* (5.4.5)*, for all* $u \in \mathcal{S}$ *we have*

$$D^\alpha \mathcal{F}^{-1}[u](x) = \mathcal{F}^{-1}[(i\xi)^\alpha u](x), \tag{5.4.12}$$
$$(-ix)^\beta \mathcal{F}^{-1}[u](x) = \mathcal{F}^{-1}[D^\beta u](x). \tag{5.4.13}$$

Proof. See Theorem 9.4.13. See Corollary 5.4.15 for another proof of (5.4.11). □

The following theorem is important because it shows that Fourier transform preserves the inner product in L^2. The action of $\mathcal{F}$ in L^p spaces is more complex; see Lemma 9.4.14 and Theorem 9.4.15 in Annex for more.

[7] For the topology induced by the seminorms s_m, in the sense $\lim_{n\to\infty} u_n = u$ in $\mathcal{S}$ implies $\lim_{n\to\infty} \mathcal{F}[u_n] = \mathcal{F}[u]$ in $\mathcal{S}$; see Definition 5.4.1.

Theorem 5.4.9 (Parseval's theorem) *Let $u, v \in \mathcal{S}$. Then $(u, v) = (\hat{u}, \hat{v})$, where $(\cdot, \cdot)$ is the inner product in L^2. Furthermore, Fourier transform $\mathcal{F}$ is extended continuously in L^2 and it defines an isometry from L^2 to itself, i.e. for all $u \in L^2$, $\|u\|_{L^2} = \|\hat{u}\|_{L^2}$.*

Proof. For $u, v \in \mathcal{S}$ we have

$$\begin{aligned}(u,v)_{L^2} &= \int_{\mathbb{R}^N} u(x)\overline{v(x)}dx = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{i(x\cdot\xi)}\hat{u}(\xi)\overline{v(x)}d\xi dx \\ &= \int_{\mathbb{R}^N} \hat{v}(\xi)\overline{\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-i(\xi\cdot x)}u(x)dx}d\xi \\ &= (\hat{u},\hat{v})_{L^2}. \end{aligned} \tag{5.4.14}$$

As $\mathcal{S} \subset L^2$ with dense inclusion, it follows that $\mathcal{F}$ extends in L^2 and (5.4.14) holds in L^2.

5.4.2 Tempered distributions and Fourier transform

The definition of Fourier transform of tempered distributions is motivated by the simple case when $T \in L^1$. Indeed, from Lemma 9.4.14 we have $\mathcal{F}[T] \in C^0 \cap L^\infty$, and with $\varphi \in \mathcal{S}$ we obtain

$$\begin{aligned}\langle \mathcal{F}[T], \varphi\rangle &= \int_{\mathbb{R}^N} \mathcal{F}[T](\xi)\varphi(\xi)d\xi = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}\varphi(\xi)\int_{\mathbb{R}^N} e^{-i(\xi\cdot x)}T(x)dxd\xi \\ &= \int_{\mathbb{R}^N} T(x)\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-i(x\cdot\xi)}\varphi(\xi)d\xi dx = \langle T, \mathcal{F}[\varphi]\rangle.\end{aligned}$$

Definition 5.4.10 *For $T \in \mathcal{S}'$, the Fourier transform of T, denoted by $\mathcal{F}[T]$, is the tempered distribution defined by*

$$\langle \mathcal{F}[T], \varphi\rangle = \langle T, \mathcal{F}[\varphi]\rangle, \quad \forall \varphi \in \mathcal{S}. \tag{5.4.15}$$

Note that Theorem 5.4.8 ensures that $\mathcal{F}[T] \in \mathcal{S}'$. Note also that in some texts $\mathcal{F}[T]$ is defined by $\langle \mathcal{F}[T], \varphi\rangle = \langle T, \mathcal{F}^{-1}[\varphi]\rangle$, which is motivated by the multiplication by the complex conjugate.

Remark 5.4.11 *Similar to Theorem 5.4.8, $\mathcal{F}$ maps continuously[8] $\mathcal{S}'$ onto itself, and formulas (5.4.4) and (5.4.5) hold for every $T \in \mathcal{S}'$. Also, $\mathcal{F}^{-1} : \mathcal{S}' \mapsto \mathcal{S}'$, the inverse of $\mathcal{F}$ in $\mathcal{S}'$, is continuous and invertible from $\mathcal{S}'$ to itself, and is given by*

$$\langle \mathcal{F}^{-1}[S], \varphi\rangle = \langle S, \mathcal{F}^{-1}[\varphi]\rangle, \quad \forall S \in \mathcal{S}'. \tag{5.4.16}$$

Furthermore, (5.4.12) and (5.4.13) hold in $\mathcal{S}'$.

[8] In the sense $\lim_{n\to\infty} u_n = u$ in $\mathcal{S}'$ implies $\lim_{n\to\infty} \mathcal{F}[u_n] = \mathcal{F}[u]$ in $\mathcal{S}'$; see Definition 5.4.3.

Remark 5.4.12 *The properties (5.4.4) and (5.4.5) are useful when solving partial differential equations. For example, consider the following PDE:*

$$L(D)u := \sum_{|\alpha|\le m} c_\alpha D^\alpha u = f \quad in \;\; \mathcal{S}',$$

with $c_\alpha \in \mathbb{C}$. By taking Fourier transform of both sides of this equation, we obtain

$$\hat{u} \sum_{|\alpha|\le m} c_\alpha (i\xi)^\alpha = \hat{f}, \quad so \quad \hat{u}(\xi) = \frac{\hat{f}}{\sum_{|\alpha|\le m} c_\alpha (i\xi)^\alpha} =: \hat{g}(\xi).$$

If $\hat{g} \in \mathcal{S}'$, so $\hat{g} = \mathcal{F}[g]$, for a certain $g \in \mathcal{S}'$, then $u = \mathcal{F}^{-1}[\hat{g}] = g$. This formula for u serves as a criteria for classifying PDEs with constant coefficients, because, given f, the existence and the regularity of u depends uniquely on the operator L (or coefficients c_α) of the PDE, or equivalently on the polynomial $\sum_{|\alpha|\le m} c_\alpha (i\xi)^\alpha$.

Example 5.4.13 *Let $a \in \mathbb{R}^N$ and $\delta_a \in \mathcal{S}'$ defined by $\langle \delta_a, \varphi\rangle = \varphi(a)$, for all $\varphi \in \mathcal{S}$. Then*

$$\mathcal{F}[\delta_a] = \frac{1}{(2\pi)^{N/2}} e^{-i(a\cdot\xi)}, \qquad \mathcal{F}[\delta_0] = \frac{1}{(2\pi)^{N/2}}, \tag{5.4.17}$$

because

$$\begin{aligned} \langle \mathcal{F}[\delta_a], \varphi\rangle &= \langle \delta_a, \mathcal{F}[\varphi]\rangle = \mathcal{F}[\varphi](a) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i(a\cdot x)} \varphi(x) dx \\ &= \left\langle \frac{e^{-i(a\cdot x)}}{(2\pi)^{N/2}}, \varphi \right\rangle. \end{aligned}$$

Example 5.4.14 *We have*

$$\mathcal{F}[1] = (2\pi)^{N/2} \delta_0. \tag{5.4.18}$$

Indeed, from the second equality of (5.4.17) we get $\mathcal{F}[(2\pi)^{N/2}\delta_0] = 1$, which implies

$$\begin{aligned} \langle \mathcal{F}[1], \varphi(x)\rangle &= \langle 1, \mathcal{F}[\varphi(x)]\rangle = \langle 1, \mathcal{F}^{-1}[\varphi(-x)]\rangle = \langle \mathcal{F}^{-1}[1], \varphi(-x)\rangle \\ &= \langle (2\pi)^{N/2}\delta_0, \varphi(-x)\rangle = (2\pi)^{N/2}\varphi(0). \end{aligned}$$

By using Example 5.4.14 we get another (simple) proof of the inverse Fourier transform $\mathcal{F}^{-1}$ in $\mathcal{S}$.

Corollary 5.4.15 $\mathcal{F}^{-1} : \mathcal{S} \mapsto \mathcal{S}$ *is given by (5.4.11).*

Proof. Indeed, let $u \in \mathcal{S}$ and $\hat{u} = \mathcal{F}[u]$. For $z \in \mathbb{R}^N$ we set

$$\mathcal{F}^{-1}[\hat{u}](z) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i(z\cdot\xi)} \hat{u}(\xi) d\xi.$$

If we prove that $\mathcal{F}^{-1}[\hat{u}](z) = u(z)$ then we have proved the formula (5.4.11). Note that

$$\begin{aligned} e^{i(\xi\cdot z)}\hat{u}(\xi) &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i(\xi\cdot(x-z))} u(x)dx = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i(\xi\cdot x)} u(x+z)dx \\ &= \mathcal{F}[u(\cdot+z)](\xi). \end{aligned}$$

Therefore from (5.4.18) we obtain

$$\begin{aligned} \mathcal{F}^{-1}[\hat{u}](z) &= \frac{1}{(2\pi)^{N/2}} \langle 1, \mathcal{F}[u(\cdot+z)](\xi)\rangle = \frac{1}{(2\pi)^{N/2}} \langle \mathcal{F}[1], u(\cdot+z))\rangle = \langle \delta_0, u(\cdot+z)\rangle \\ &= u(z). \end{aligned}$$

Hence, $\mathcal{F}^{-1}[\mathcal{F}[u]] = u$. In a similar way one can prove $\mathcal{F}[\mathcal{F}^{-1}[U]] = U$, for all $U \in \mathcal{S}$.

Example 5.4.16 *Let $x \in \mathbb{R}$. Clearly $x \in \mathcal{S}'$ and from (5.4.4), (5.4.5) we get*

$$F[x] = \frac{1}{-i}\mathcal{F}[(-ix)1] = i\frac{d}{d\xi}\mathcal{F}[1] = i\frac{d}{d\xi}\left((2\pi)^{1/2}\delta_0\right) = i\sqrt{2\pi}\delta_0'.$$

Example 5.4.17 *Let us find the solutions to*

$$-\Delta u = \delta_0 \ \ in \ \ \mathcal{S}', \ \ N \geq 1. \tag{5.4.19}$$

First we note that $u \in C^\infty(\mathbb{R}^N\backslash\{0\})$; see Problem 5.17. Also, u must be radially symmetric in $\mathbb{R}^N\backslash\{0\}$. Indeed, taking Fourier transform of (5.4.19) gives $|\xi|^2\hat{u} = \dfrac{1}{(2\pi)^{N/2}}$. So $\hat{u}$ is radially symmetric. Then for every rotation matrix $R \in \mathbb{R}^{N\cdot N}$, ${}^tR = R^{-1}$, and $\varphi \in \mathcal{D}(\mathbb{R}^N)$ with $\mathrm{supp}(\varphi) \cap \{0\} = \emptyset$ we have

$$\begin{aligned} \langle u, \varphi\rangle &= \langle \hat{u}, \mathcal{F}^{-1}[\varphi]\rangle = \langle \hat{u}\circ R, \mathcal{F}^{-1}[\varphi]\rangle = \langle \hat{u}, (\mathcal{F}^{-1}[\varphi])\circ({}^tR)\rangle \\ &\qquad = \langle \hat{u}, \mathcal{F}^{-1}[\varphi\circ({}^tR)]\rangle \\ &\qquad = \langle \mathcal{F}^{-1}[\hat{u}], \varphi\circ({}^tR)\rangle \\ &= \langle u\circ R, \varphi\rangle, \end{aligned}$$

which implies $u(x) = u(R\cdot x)$, $|x| \neq 0$. So $u = u(r)$, $r = |x|$. From the formula of Δ in spherical coordinates, we get

$$\Delta u(x) = \frac{1}{r^{N-1}}\partial_r(r^{N-1}\partial_r u(r)) = 0 \quad in \ \ \mathbb{R}^N\backslash\{0\}.$$

Integrating this equation and neglecting the constant terms implies

$$\begin{aligned}
&N = 1: \quad u(x) = C|x|, \quad C \in \mathbb{R},\\
&N = 2: \quad u(x) = C \ln |x|,\\
&N \geq 3: \quad u(x) = C|x|^{2-N}.
\end{aligned}$$

The constant C is found by taking a particular test function in $\langle -\Delta u, \varphi\rangle = \langle \delta_0, \varphi\rangle$, for example $\varphi(x) = e^{-|x|}$ (which is allowed because δ_0 is a distribution of order zero). In the case $N \geq 3$ we get

$$\begin{aligned}
1 &= \langle \delta_0, \varphi\rangle = \langle -\Delta u, \varphi\rangle = -\int_{S^{N-1}}\int_0^\infty C r^{2-N} r^{N-1} r^{1-N} \partial_r (r^{N-1}\partial_r e^{-r}) dr d\sigma\\
&= C|S_N| \int_0^\infty ((N-1)e^{-r} - re^{-r}) dr\\
&= C(N-2)|S_N|,
\end{aligned}$$

where $|S_N|$ is the area of the unit sphere in $\mathbb{R}^N$. Hence $C = \dfrac{1}{(N-2)|S_N|}$.
In general, for arbitrary dimension $N \geq 1$, the unique (up to a constant) solution in $\mathcal{S}'$ of (5.4.19) is given by

$$E(x) = \begin{cases} -\dfrac{1}{2}|x|, & N = 1,\\ -\dfrac{1}{2\pi}\ln|x|, & N = 2,\\ \dfrac{1}{(N-2)|S_N|}\dfrac{1}{|x|^{N-2}}, & N \geq 3. \end{cases} \tag{5.4.20}$$

The distribution E is called a "fundamental solution to $-\Delta$". Using it one shows that the solution to

$$-\Delta u = f, \; f \in \mathcal{D}'$$

is given by $u = G * f$. For similar results for general linear operators with constant coefficients, see Definition 5.3.4 and Theorem 5.3.6.

Example 5.4.18 *Let us find a fundamental solution G to the heat equation solving (5.3.7); see Example 5.3.7. By taking Fourier transform of the first equation of (5.3.7) with respect to x and then using (5.4.10) with $a = t$ gives*

$$\begin{aligned}
\partial_t \hat{G} + \xi^2 \hat{G} &= 0, \quad \textit{so} \quad \hat{G}(\xi, t) = Ce^{-t\xi^2} \quad \textit{and}\\
G(x,t) &= \frac{C}{\sqrt{2t}} e^{-\frac{1}{4}\frac{x^2}{t}}, \quad C \in \mathbb{R}.
\end{aligned}$$

The constant C is chosen such that $\lim_{t\to 0}\int_{\mathbb{R}} G(x,t)\varphi(x)ds = \varphi(0)$. By changing the variable and using (5.4.7), one finds easily $C = \frac{1}{\sqrt{2\pi}}$, which leads to (5.3.10).

Similarly, we can find the fundamental solution G to wave equation solving (5.3.11); see Example 5.3.8. If $T = \partial_t G$ then T satisfies

$$\partial_{tt}T(\cdot,t) - \Delta T(\cdot,t) = 0 \ \text{ for } \ t > 0, \quad T(\cdot,0) = \delta_0.$$

By taking Fourier transform of the first equation above with respect to x and then using (5.4.17), we get

$$\begin{aligned}
\hat{T}_{tt} + \xi^2\hat{T} &= 0; \quad \text{so}\\
\hat{T}(\xi,t) &= Ae^{i\xi t} + Be^{-i\xi t} \quad \text{with } \ A,\ B \in \mathbb{C} \quad \text{and therefore}\\
T(x,t) &= \sqrt{2\pi}(A\delta_0(x+t) + B\delta_0(x-t)).
\end{aligned}$$

We choose $A = B = \dfrac{1}{2}\dfrac{1}{\sqrt{2\pi}}$ so that $T(\cdot,0) = \delta_0$ and $T(x,t) = \dfrac{1}{2}(\delta_0(x+t) + \delta_0(x-t))$. From $\partial_t G(x,t) = T(x,t)$ and $G(\cdot,0) = 0$ it follows that

$$G(x,t) = \frac{1}{2}(H(x+t) - H(x-t)) = \frac{1}{2}H(t - |x|).$$

Problems

Problem 5.1 *Prove that $\rho \in \mathcal{D}(\mathbb{R}^N)$, where*

$$\forall x \in \mathbb{R}^N, \quad \rho(x) = \begin{cases} e^{\frac{1}{|x|^2-1}}, & |x| < 1,\\ 0, & |x| \geq 1.\end{cases}$$

Problem 5.2 *Let $\Omega \subset \mathbb{R}^N$ open, $u,\ v \in C^0(\Omega)$, $u \neq v$, and T_u, T_v be their corresponding distributions in $\mathcal{D}'(\Omega)$. Show that $T_u \neq T_v$.*

Problem 5.3 *Find which of the following maps in $\mathcal{D}$ define a distribution in $\mathbb{R}$:*

a) $\langle T,\varphi\rangle = \sum_{k=1}^{\infty}\varphi(k)$, b) $\langle T,\varphi\rangle = \sum_{k=1}^{\infty}\varphi^2(k)$,

c) $\langle T,\varphi\rangle = \sum_{k=1}^{\infty}\varphi^{(k)}(1)$, d) $\langle T,\varphi\rangle = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\varphi^{(k)}(k)$.

Problem 5.4 *Prove that if $T \in \mathcal{D}'(\Omega)$ then $D^\alpha T \in \mathcal{D}'(\Omega)$.*

Problem 5.5 *Find a distribution in $\mathbb{R}$ which is not of finite order.*

Problem 5.6 *Let $\Omega \subset \mathbb{R}^N$ be open and $T \in \mathcal{D}'(\Omega)$. Assume $T \geq 0$, i.e. $\langle T, \varphi \rangle \geq 0$ for all real-valued φ, $0 \leq \varphi \in \mathcal{D}(\Omega)$. Prove that T is a distribution of order zero (a measure).*

Problem 5.7 *Let $n \in \mathbb{N}_0$, $T \in \mathcal{D}'(\mathbb{R})$ and consider the equation $x^{n+1}T = 0$.*

i) Assume $n = 0$. Prove that $T = c_0\delta_0$, for arbitrary $c_0 \in \mathbb{C}$.

ii) Assume $n \geq 1$. Prove that $T = \sum_{k=0}^{n} c_k \delta_0^{(k)}$, $c_k \in \mathbb{C}$.

Problem 5.8 *Prove that if $T \in \mathcal{D}'(\mathbb{R})$ and $\operatorname{supp}(T) = \{0\}$ then there exists $m \in \mathbb{N}_0$ and $c_\alpha \in \mathbb{C}$, $|\alpha| \leq m$, such that*[9] $T = \displaystyle\sum_{k=0}^{m} c_k \delta_0^{(k)}$.

Problem 5.9 *Let $T_n := n\chi_{(0,\frac{1}{n})}$. Prove each of the following.*

i) T_n has a limit in $\mathcal{D}'(\mathbb{R})$.

ii) T_n^2 does not have a limit in $\mathcal{D}'(\mathbb{R})$.

iii) $T_n^2 - n\delta_0$ has a limit in $\mathcal{D}'(\mathbb{R})$.

Problem 5.10 *Let $u_n = \dfrac{\sin(nx)}{\sqrt{n}}$. Find the limit of u_n and u_n' in $\mathcal{D}'(\mathbb{R})$.*

Problem 5.11 *Find the limit in $\mathcal{D}'(\mathbb{R})$ of the following distributions:*

a) $T_n = n^{1/2}e^{-nx^2}$, *b)* $T_n = e^{n^2}\delta_0(x-n)$,

c) $T_n = \dfrac{\sin(nx)}{nx}$, *d)* $T_n = cos(nx) + i(\cos(nx))^2$, $i^2 = -1$.

Problem 5.12 (Leibniz rule) *Let $k \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, $\Omega \in \mathbb{R}^N$ open. Prove that $\langle kT, \varphi \rangle = \langle T, k\varphi \rangle$ defines a distribution $kT \in \mathcal{D}'(\Omega)$ and*

$$\partial_i(kT) = k\partial_i T + (\partial_i k)T, \quad i = 1, \ldots, N, \tag{5.4.21}$$

$$D^\alpha(kT) = \sum_{0 \leq \beta \leq \alpha} C_\beta^\alpha D^\beta k D^{\alpha-\beta} T, \quad C_\beta^\alpha = \frac{\alpha!}{\beta!(\alpha-\beta)!}, \quad \forall \alpha \in \mathbb{N}_0^N. \tag{5.4.22}$$

Problem 5.13 *Let $u = |x|$, $x \in \mathbb{R}$. Find u' and u'' in $\mathcal{D}'(\mathbb{R})$.*

Problem 5.14 *Let $u \in C^1(\mathbb{R}\backslash\{x_n,\, n \in \mathbb{Z}\})$ with u having left and right limits at every x_n. Write the distribution u'.*

[9] Note that the result holds in dimension N: if $T \in \mathcal{D}'(\mathbb{R}^N)$ and $\operatorname{supp}(T) = \{0\}$ then there exist $m \in \mathbb{N}_0$ and $c_\alpha \in \mathbb{C}$, $|\alpha| \leq m$, such that $T = \displaystyle\sum_{|\alpha| \leq m} c_\alpha D^\alpha \delta_0$.

Problem 5.15 *Let* $u(x) = e^{-|x|}$, $x \in \mathbb{R}$. *Find* $-\Delta u + u$ *in* $\mathscr{D}'(\mathbb{R})$.

Problem 5.16 *Let* $T \in \mathscr{D}'(\mathbb{R})$, $h \in \mathbb{R}$, $\delta > 0$ *and define* T_h *by* $\langle T_h, \varphi \rangle = \langle T, \varphi(\cdot + h) \rangle$. *Show that* $T_h \in \mathscr{D}'(\mathbb{R})$. *Furthermore, show that if* $T_h = T$ *for all* $|h| < \delta$, *then* T *is constant.*

Problem 5.17 *Let* $u \in \mathscr{D}'(\mathbb{R}^N)$, $\Delta u = 0$ *in* $\mathscr{D}'(\mathbb{R}^N)$. *Prove that* $u \in C^\infty(\mathbb{R}^N)$.

Problem 5.18 *Let* $f \in \mathscr{D}'(\mathbb{R})$ *and* $k \in C^\infty(\mathbb{R})$. *Find the (general) solution* $T \in \mathscr{D}'(\mathbb{R})$ *to* $T' = f$ *and of* $T' + kT = f$.

Problem 5.19 *Find* $\mathscr{F}[u]$, *where* $u = u(x)$, $x \in \mathbb{R}$, *is given by*

a) $u = \sin(x)$, *b)* $u = \dfrac{1}{1+x^2}$, *c)* $u = \dfrac{\sin x}{1+x^2}$,

d) $u = \sin(x^2)$, *e)* $u = H(x)$, *f)* $u = |x|$,

where H *is the Heaviside function.*

Problem 5.20 *Find the general solution to* $y'' \pm y = 0$ *in* $\mathscr{S}'(\mathbb{R})$.[10]

Problem 5.21 *Let* u *be given by* (5.3.8) *with* $G(x,t) = \dfrac{1}{(4\pi t)^{N/2}} e^{-\frac{1}{4}\frac{|x|^2}{t}}$, $x \in \mathbb{R}^N$, $t > 0$. *Show that* u *solves* (5.3.9) *and* G *solves* (5.3.7) *in dimension* N.

Problem 5.22 *Assume* $u \in \mathscr{D}$ *and* $\mathscr{F}[u] \in \mathscr{D}$. *Prove that* $u = 0$.

Problem 5.23 *Prove that Fourier transform in* $L^1(\mathbb{R}^N)$ *is injective.*

[10] In general, an equation $\sum_{k=0}^{n} c_k y^{(k)} = 0$ is transformed to $P(\xi)\hat{y} = 0$, with P a polynomial. By using the roots ξ_i of $P(\xi) = 0$, each with multiplicity m_i, it leads to $\hat{y} = \sum_i \sum_{j=0}^{m_i - 1} a_{i,j} \delta_0^{(j)}(\xi - \xi_i)$, which after applying $\mathscr{F}^{-1}$ gives y.

6. Sobolev spaces

Sobolev spaces were introduced by Sergei Lvovich Sobolev in the 1930s, and have since been widely embraced and developed by other mathematicians. Sobolev spaces represent a natural functional framework to describe a rich variety of real-world problems, and they provide solutions to a large number of PDEs. Mathematically, they provide elegant tools for studying PDEs because they have rich properties in terms of approximation, compactness, and boundary values.

In order to avoid technicalities related to the analysis of general $W^{s,p}(\Omega)$ spaces, $s \in \mathbb{R}$, $p \in [1, \infty]$, we will focus mainly on $H^s(\Omega) = W^{s,2}(\Omega)$ spaces, $s > 0$. The reason for this is that the analysis of $H^s(\Omega)$ spaces is greatly simplified by the use of Fourier transform, as well as the fact that $H^s(\Omega)$ spaces are Hilbert spaces.

For more on $W^{s,p}(\Omega)$ spaces in connection with the development of this chapter, see section 9.5 in Annex, and for an extensive treatment of Sobolev spaces, see classical books such as [1, 5, 14, 33, 40, 44, 51].

6.1 Definitions and some first properties

The reader is invited to review chapter 1.

Definition 6.1.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set, $k \in \mathbb{N}$, $p \in [1, \infty]$.*

1) For $p \in [1, \infty)$, we set

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega),\ \forall \alpha \in \mathbb{N}_0^N,\ |\alpha| \leq k\}, \tag{6.1.1}$$

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}, \quad |u|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| = k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}, \tag{6.1.2}$$

and for $p = \infty$, set

$$W^{k,\infty}(\Omega) = \{u \in L^\infty(\Omega),\ D^\alpha u \in L^\infty(\Omega),\ \forall \alpha \in \mathbb{N}_0^N,\ |\alpha| \leq k\}, \tag{6.1.3}$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}, \quad |u|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| = k} \|D^\alpha u\|_{L^\infty(\Omega)}. \tag{6.1.4}$$

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6_6

2) *The space $W^{k,p}(\Omega)$ is called a "$W^{k,p}$ Sobolev space in Ω". The integer k is called an "order of $W^{k,p}(\Omega)$", and $k-\frac{N}{p}$ is called a "differential dimension of $W^{k,p}(\Omega)$". The quantity $\|u\|_{W^{k,p}(\Omega)}$, resp. $|u|_{W^{k,p}(\Omega)}$, is called a "norm, resp. seminorm of u in $W^{k,p}(\Omega)$".*

3) *For $p=2$, we write $H^k(\Omega)$ instead of $W^{k,2}(\Omega)$. In $H^k(\Omega)$, we define an inner product*

$$\forall u, v \in H^k(\Omega): \quad (u,v)_{H^k(\Omega)} := \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u D^\alpha v dx. \tag{6.1.5}$$

4) *For $p \in [1,\infty)$, sometimes we use the other equivalent norm and seminorm:*

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p(\Omega)}, \qquad |u|_{W^{k,p}(\Omega)} = \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\Omega)}. \tag{6.1.6}$$

Also, when $\Omega=\mathbb{R}^N$, we will write $W^{k,p}$, H^k, instead of $W^{k,p}(\mathbb{R}^N)$, $H^k(\mathbb{R}^N)$.

5) *If $E(\Omega)$ is any of the spaces above, we define $E_{loc}(\Omega)=\{u\in E(\omega),\ \ \forall\omega\Subset\Omega\}$.*

For $u \in L^p(\Omega)$, we have $D^\alpha u \in \mathscr{D}'(\Omega)$. Definition 6.1.1 requires $D^\alpha u$ to be a $L^p(\Omega)$ function. Furthermore, functions of $W^{k,p}(\Omega)$ are defined almost everywhere in Ω. More precisely, the elements of $W^{k,p}(\Omega)$ are classes of equivalences defined by the relation "*$u=v$ if and only if (iff) $u=v$ a.e. in Ω, $u,v\in W^{k,p}(\Omega)$*"; see Problem 6.1. The elements of the same equivalent class have the same $W^{k,p}(\Omega)$ norm. This is the reason why throughout this text we will not make the distinction between a function of $W^{k,p}(\Omega)$ and its equivalent class.

The following result is classical. The proof of completeness is simple and based on the completeness of $L^p(\Omega)$ spaces, while the separability and reflexivity require some results from Functional Analysis. The proof can be found in many books treating Sobolev spaces; see, for example, [1, 5, 14].

Theorem 6.1.2 *Let $k\in\mathbb{N}$ and $\Omega\subset\mathbb{R}^N$ be open. $W^{k,p}(\Omega)$ equipped with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is a Banach space for all $p\in[1,\infty]$. $W^{k,p}(\Omega)$ is separable for all $p\in[1,\infty)$, and reflexive for all $p\in(1,\infty)$.*

Example 6.1.3 *Let $u(x)=|x|^q$, $\Omega=B(0,1)\subset\mathbb{R}^N$, $q\neq 0$.*

i) *We have $u\in W^{1,p}(B)$ iff $p(q-1)>-N$. Indeed, considering spherical coordinates (r,θ) and if $S_N=\{x\in\mathbb{R}^N,\ |x|=1\}$ is the unit sphere in $\mathbb{R}^N$, we have*

$$\int_B |u|^p dx \;=\; \int_{S_N}\int_0^1 r^{N-1}r^{pq}drd\theta = |S_N|\int_0^1 r^{N-1+pq}dr.$$

Hence $u \in L^p(B)$ iff $pq > -N$. For the derivatives, note that $\partial_i u = q|x|^{q-2}x_i$ and

$$\int_B |\partial_i u|^p dx = q^p \int_{S_N} \int_0^1 r^{N-1} r^{p(q-2)} |x_i|^p dr d\theta \sim \int_0^1 r^{N-1+p(q-1)} dr,$$

where $a \sim b$ means $C_1 a \le b \le C_2 a$, for certain $C_1, C_2 > 0$. Hence $\partial_i u \in L^p(B)$ iff $p(q-1) > -N$, which proves the claim.

ii) In general, for $k \in \mathbb{N}$, $p \in [1,\infty]$, $u \in W^{k,p}(B)$, iff $p(q-k) > -N$, or equivalently $q > k - \frac{N}{p}$.

iii) Similar calculus show that $k \in \mathbb{N}$, $p \in [1,\infty]$, $u \in W^{k,p}(\overline{B}^c)$, iff $q < k - \frac{N}{p}$.

Example 6.1.4 *Let $u(x) = \ln|x|$, $x \in \mathbb{R}^N$, $\Omega = B(0,1) \subset \mathbb{R}^N$. Then as in the example above, we can prove that $u \in W^{1,p}(\Omega)$ iff $p < N$.*

Example 6.1.5 *Let $u(x) = \mathbb{1}_{(0,1)}(x)$, $x \in (-1,1)$. Then $u \notin W^{1,p}(-1,1)$, $p \in [1,\infty]$. Indeed, let $u' \in \mathcal{D}'(-1,1)$ be the distributional derivative of u. Then $u' = \delta_0$ and $\delta_0 \notin L^p(-1,1)$ for all $p \in [1,\infty]$; see Example 5.2.8.*

Example 6.1.6 *Let $\Omega = A \cup \gamma \cup B$, where A, B are open bounded sets, and assume $\gamma = \partial A \cap \partial B \neq \emptyset$ is a smooth boundary. Set $u = \alpha \mathbb{1}_A + \beta \mathbb{1}_B$ with $\alpha, \beta \in \mathbb{R}$. If $\alpha \neq \beta$ then $u \notin W^{1,p}(\Omega)$, $p \in [1,\infty]$.*

We prove the claim by contradiction, so assume $u \in W^{1,p}(\Omega)$, $p \in [1,\infty]$. Also, for simplicity we consider $A = (-1,0) \times (0,1)$, $B = (0,1) \times (0,1)$, $\gamma = \{0\} \times (0,1)$, and $p = 2$. So $u(x_1,x_2) = (\beta - \alpha)H(x_1) + \alpha$. Then for every $\varphi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned} \langle \partial_2 u, \varphi \rangle &= 0, \\ \langle \partial_1 u, \varphi \rangle &= -\langle u, \partial_1 \varphi \rangle = \int_0^1 (\beta - \alpha)\varphi(0,x_2) dx_2 =: \langle (\beta-\alpha)\delta_\gamma, \varphi \rangle, \end{aligned}$$

with $\delta_\gamma \in L^2(\Omega)$. Note also that $\delta_\gamma \neq 0$ because if not we get $\nabla u = 0$ and then u is constant (see Example 5.2.21), which is impossible because $\alpha \neq \beta$.

From the density of $\mathcal{D}(\Omega)$ in $L^2(\Omega)$, there exists (φ_n) in $\mathcal{D}(\Omega)$ converging to δ_γ in $L^2(\Omega)$. WLOG we may assume that the convergence of (φ_n) is also pointwise in Ω and $|\varphi_n| \le v$ a.e. Ω for all n, for a certain function $v \in L^2(\Omega)$; see Theorem 1.3.7. Set $\psi_n(x_1,x_2) = (1 - \eta_n(x_1))\varphi_n(x_1,x_2)$, where $\eta_n(x_1) = \eta(nx_1)$ and η is a $\{[-1,1],(-2,2)\}$ cut-off function. It follows that $\lim_{n\to\infty} \|\psi_n - \delta_\gamma\|_{L^2(\Omega)} = 0$, because

$$\begin{aligned} \|\psi_n - \varphi_n\|^2_{L^2(\Omega)} &= \int_\Omega |\eta_n(x_1)\varphi_n(x)|^2 dx = \int_{\{|x_1|<2/n\}} |\eta_n(x_1)\varphi_n(x)|^2 dx \\ &\underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

where for the limit we used Lebesgue DCT. It implies

$$0 \neq \|\delta_\gamma\|^2_{L^2(\Omega)} = \lim_{n\to\infty} \int_\Omega \delta_\gamma \psi_n dx = \lim_{n\to\infty} \langle \delta_\gamma, \psi_n \rangle = \lim_{n\to\infty} \int_0^1 \psi_n(0,x_2) dx_2 = 0,$$

because $\psi_n(0,x_2) = 0$. The contradiction shows that $u \notin W^{1,2}(\Omega)$.

6.1.1 Density of $\mathscr{D}$ in $W^{k,p}$

For $k \in \mathbb{N}$ and $p \in [1, \infty)$, we have $\mathscr{D} \subset W^{k,p}$. In the following theorem we show that this embedding is dense. The theorem involves two steps, regularization and truncation, which are frequently used when working with Sobolev space functions. We emphasize that the following theorems 6.1.7 and 6.1.8 do not hold for $p = \infty$.

Theorem 6.1.7 (Density of $\mathscr{D}$ and $\mathscr{S}$ in $W^{k,p}$) *Let $k \in \mathbb{N}$, $p \in [1, \infty)$. Then the spaces $\mathscr{D}$ and $\mathscr{S}$ are included densely in $W^{k,p}$, $p \in [1, \infty)$. We write $\mathscr{D} \overset{d}{\hookrightarrow} W^{k,p}$, $\mathscr{S} \overset{d}{\hookrightarrow} W^{k,p}$.*

Proof. The proof is made in two steps.
Regularization. Let (ρ_n) be a mollifier sequence, and for $u \in W^{k,p}$ set $v_n = u * \rho_n$. For $\alpha \in \mathbb{N}_0^N$, $|\alpha| \leq k$, from Theorem 5.3.3 we have

$$D^\alpha v = D^\alpha (u * \rho_n) = D^\alpha u * \rho_n, \tag{6.1.7}$$

which in combination with i), Theorem 5.2.5, implies $v_n \in C^\infty \cap L^p$ and $\lim\limits_{n\to\infty} v_n = u$ in $W^{k,p}$. In general, (v_n) is not in $\mathscr{D}$ and to generate a sequence in $\mathscr{D}$ we proceed as follows.
Truncation. Let η be a $\{\overline{B}(0,1), B(0,2)\}$ cut-off function and set $\eta_n(x) = \eta(x/n)$, $u_n = \eta_n v_n$. Then $u_n \in \mathscr{D}$ and $\lim_{n\to\infty} u_n = u$ in $W^{k,p}$ because

$$\begin{aligned} \|D^\alpha u_n - D^\alpha u\|_{L^p} \leq{}& \|\eta_n (D^\alpha v_n - D^\alpha u)\|_{L^p} + \|(1 - \eta_n) D^\alpha u\|_{L^p} \\ &+ \sum_{0<\beta\leq\alpha} C_\alpha^\beta \|D^\beta \eta_n D^{\alpha-\beta} v_n\|_{L^p}, \end{aligned}$$

where we used the result of Problem 5.12. The first integral tends to zero from the construction of v_n, and the second integral tends to zero from Lebesgue DCT. For the third integral, one notices that $D^\beta \eta_n(x) = n^{-|\beta|} D^\beta \eta(x/n)$, which implies that this integral also tends to zero.

The claim for $\mathscr{S}$ follows from the fact that given $\varphi \in \mathscr{S}$, if $\varphi_n = \eta_n \varphi$ then $\lim_{n\to\infty} \varphi_n = \varphi$ in $W^{k,p}$, which is proven similarly as the convergence of u_n to u above. □

The following result, even though it only provides the density of $\mathscr{D}(\Omega)$ in $L^p(\Omega)$ and in $W^{k,p}_{loc}(\Omega)$, is useful in many applications because it does not depend on the regularity of Ω.

Theorem 6.1.8 (Friedrich's theorem) *Let $k \in \mathbb{N}$, $p \in [1, \infty)$, and $\Omega \subsetneq \mathbb{R}^N$. For every $u \in W^{k,p}(\Omega)$, there exists (u_n) in $\mathscr{D}(\Omega)$ such that $\lim_{n\to\infty} u_n = u$ in $L^p(\Omega)$ and $\lim_{n\to\infty} u_n = u$ in $W^{k,p}_{loc}(\Omega)$, i.e.*

$$\lim_{n\to\infty} u_n = u \ \ in \ \ L^p(\Omega) \ \ and \ \ \lim_{n\to\infty} u_n = u \ \ in \ \ W^{k,p}(\omega), \ \ \forall \omega \Subset \Omega. \tag{6.1.8}$$

Proof. The proof will be carried out in two steps.
i) Let $\omega \subset \overline{\omega} =: K \subset G \Subset \Omega$, with ω and G open, and let $\eta \in \mathscr{D}(\mathbb{R}^N)$ be a $\{K, G\}$

cut-off function as given by Theorem 1.2.2. Note that $\eta = 1$ in K, $\mathrm{supp}(\eta) \subset G$ and $dist(\mathrm{supp}(\varphi), G^c) > 0$.

Let U be the extension in $\mathbb{R}^N$ of u by zero and set $V = \eta U$. From Lemma 9.5.7, we have $V \in W^{k,p}(\mathbb{R}^N)$. Set $V_n = \rho_n * V$. As $D^\alpha(V * \rho_n) = D^\alpha V * \rho_n$, see (6.1.7), from Theorem 5.2.5 it follows $\lim_{n\to\infty} V_n = V$ in $W^{k,p}(\mathbb{R}^N)$. Furthermore

$$V_n(x) \;=\; \int_{\mathrm{supp}(\eta)} \rho_n(x-y)\eta(y)u(y)dy = \int_{\mathrm{supp}(\eta)\cap B(x,1/n)} \rho_n(x-y)\eta(y)u(y)dy.$$

It follows that $V_n \in C^\infty(\mathbb{R}^N)$ and if $1/n < dist(\mathrm{supp}(\eta), G^c)$ then $V_n(x) = 0$ for $x \in G^c$. Hence, if $u_n = V_n|_\Omega$ we have $u_n \in \mathcal{D}(\Omega)$. Also, as $\eta = 1$ in K, from $\lim_{n\to\infty} V_n = V$ in $W^{k,p}(\mathbb{R}^N)$ it follows $\lim_{n\to\infty} u_n = u$ in $W^{k,p}(\omega)$. Therefore, we have proved that for every $\omega \Subset G \Subset \Omega$ and for every $n \in \mathbb{N}$, there exists $u_n \in \mathcal{D}(\Omega)$,

$$\|u_n - \eta u\|_{L^p(\Omega)} + \|u_n - u\|_{W^{k,p}(\omega)} < 1/n.$$

ii) Now we will repeat i) for an increasing sequence of domains like ω and G. Namely, for $n \in \mathbb{N}$ set $O_n = \{x \in \Omega,\ d(x, \partial\Omega) < 1/n,\ |x| < n\}$. Note that O_n is open, $\overline{O}_n$ is compact, $O_n \subset O_{n+1}$, and $\cup_{n\in\mathbb{N}} O_n = \Omega$.

For every $n \in \mathbb{N}$, from part i) with $\omega = O_n$ and $G = \overline{O}_{n+1}$, there exists $u_n \in \mathcal{D}(\Omega)$ and $\eta_n \in \mathcal{D}(\Omega)$, a $\{\overline{O}_n, O_{n+1}\}$ cut-off function, such that

$$\|u_n - \eta_n u\|_{L^p(\Omega)} + \|u_n - u\|_{W^{k,p}(O_n)} < 1/n.$$

Then (u_n) is the desired sequence. Indeed, from Lebesgue's dominated convergence theorem we have

$$\|u_n - u\|^p_{L^p(\Omega)} \le C\left(\int_\Omega |u_n - \eta_n u|^p + \int_\Omega |1-\eta_n|^p |u|^p\right) \to 0 \;\; as \;\; n \to \infty.$$

Finally, for a given $\omega \Subset \Omega$ there exists n_ω such that $\omega \subset O_n$ for all $n > n_\omega$. Therefore for such n, we have

$$\|u_n - u\|_{W^{k,p}(\omega)} \le \|u_n - u\|_{W^{k,p}(G_n)} < 1/n,$$

which completes the proof. □

Using a similar technique as above, one can prove the following result; see, for example, [1, 5, 14].

Theorem 6.1.9 *Let $k \in \mathbb{N}$, $p \in [1,\infty)$ and $\Omega \subset \mathbb{R}^N$ be open. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

6.1.2 Some applications

In this section, we present some applications of Theorem 6.1.8, which show that we can make a number of operations with Sobolev space functions as if they were usual classical functions.

We note that in general, if $u,\ v \in W^{k,p}(\Omega)$ then $uv \notin W^{k,p}(\Omega)$. For example, let $\Omega = B(0,1)$, and $u = |x|^q$, $q \in \mathbb{R}$. Then from Example 6.1.3, $u \in W^{k,p}(\Omega)$ if and only if $q > k - \frac{N}{p}$. We can choose $q = -\frac{1}{2}$ and $p \in [1,\infty]$ such that $k - \frac{N}{p} = -1$ (for example $p = \frac{N}{k+1}$), so that $u \in W^{k,p}(\Omega)$. However, if $v = u$ then $uv = |x|^{2q} \notin W^{k,p}(\Omega)$ because $2q = -1 \not> -1 = k - \frac{N}{p}$. The following proposition gives a sufficient condition for the product uv to be in $W^{1,p}(\Omega)$.

Proposition 6.1.10 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $p \in [1,\infty]$. Then $uv \in W^{1,p}(\Omega)$ and for all $|\alpha| = 1$, we have*

$$D^\alpha(uv) = vD^\alpha u + uD^\alpha v. \tag{6.1.9}$$

Proof. Clearly $uv, vD^\alpha u, uD^\alpha v \in L^p(\Omega)$. So it is enough to prove (6.1.9). We assume first $p \in [1,\infty)$. For $\varphi \in \mathcal{D}(\Omega)$, there exists $\omega \Subset \Omega$ such that $\mathrm{supp}(\varphi) \subset \omega$. Let (u_n), resp. (v_n), be a sequence in $\mathcal{D}(\Omega)$ satisfying Theorem 6.1.8, associated with u, resp. v, and such that furthermore u_n, resp. v_n, converges a.e. and in $W^{1,p}(\omega)$ to u, resp. v (see Theorem 1.3.7). Then for $\alpha \in \mathbb{N}_0^N$, $|\alpha| = 1$, we get

$$\begin{aligned}
\langle D^\alpha(uv), \varphi\rangle &= -\int_\Omega uvD^\alpha\varphi dx \\
&= -\lim_{n\to\infty}\int_\Omega u_n v_n D^\alpha\varphi = \lim_{n\to\infty}\int_\omega \varphi(D^\alpha u_n v_n + u_n D^\alpha v_n) \\
&= \int_\Omega (D^\alpha uv + uD^\alpha v)\varphi,
\end{aligned}$$

where when passing to the limit we used Lebesgue's dominated convergence theorem.

In the case $p = \infty$, given $\varphi \in \mathcal{D}(\Omega)$ we choose an open bounded set ω, $\mathrm{supp}(\varphi) \subset \omega \Subset \Omega$. We can apply the case $p \in [1,\infty)$ in ω, which proves (6.1.9) and completes the proof. $\square$

We leave as an exercise the proof of the following proposition.

Proposition 6.1.11 *Let $u \in W^{k,p}(\Omega)$ and $v \in W^{k,\infty}(\Omega)$. Then $uv \in W^{k,p}(\Omega)$ and for all $|\alpha| \le k$, we have*

$$D^\alpha(uv) = \sum_{\beta\le\alpha} C_\beta^\alpha D^\beta u D^{\alpha-\beta} v, \quad C_\beta^\alpha = \frac{\alpha!}{\beta!(\alpha-\beta)!}. \tag{6.1.10}$$

The following two propositions show how the operation of composition holds in the context of Sobolev spaces.

Proposition 6.1.12 *Let $\Omega \subset \mathbb{R}^N$ be an open set, $h \in C^1(\mathbb{R})$, $h' \in L^\infty(\mathbb{R})$, and $h(0) = 0$. If $u \in W^{1,p}(\Omega)$, $p \in [1, \infty]$ then $h \circ u \in W^{1,p}(\Omega)$ and*

$$h \circ u \in W^{1,p}(\Omega), \qquad D^\alpha(h \circ u) = h'(u) D^\alpha u, \;\; \forall \alpha \in \mathbb{N}_0^N, \;\; |\alpha| = 1. \tag{6.1.11}$$

Proof. As $|h(t)| \le \|h'\|_{L^\infty(\mathbb{R})} |t|$, it follows that $h \circ u$, $h'(u) D^\alpha u \in L^p(\Omega)$. It is enough then to prove the equality in (6.1.11).

We consider first the case $p \in [1, \infty)$. Let ω be an open set, $\omega \Subset \Omega$. From Theorem 6.1.8, there exists $u_n \in \mathcal{D}(\Omega)$ such that $\lim_{n\to\infty} u_n = u$ in $L^p(\Omega)$ and $\lim_{n\to\infty} \nabla u_n = \nabla u$ in $L^p(\omega)$. Also, without restriction we may assume that $u_n \to u$ pointwise in Ω, and $\nabla u_n \to \nabla u$ pointwise in ω; see Theorem 1.3.7.

From the assumption for h and the convergence of u_n, we have $\lim\limits_{n\to\infty} h(u_n) = h(u)$ in $L^p(\Omega)$. Using Lebesgue DCT and the convergence a.e. of u_n and of ∇u_n yields $\lim\limits_{n\to\infty} h'(u_n) = h'(u)$ and $\lim\limits_{n\to\infty} h'(u_n) D^\alpha u_n = h'(u) D^\alpha u$ in $L^p(\omega)$. Then we get

$$\begin{aligned}
\langle D^\alpha(h \circ u), \varphi \rangle &= -\langle h \circ u, D^\alpha \varphi \rangle \\
&= -\lim_{n\to\infty} \langle h \circ u_n, D^\alpha \varphi \rangle = \lim_{n\to\infty} \langle D^\alpha(h \circ u_n), \varphi \rangle = \lim_{n\to\infty} \langle h'(u_n) D^\alpha u_n, \varphi \rangle \\
&= \langle h'(u) D^\alpha u, \varphi \rangle,
\end{aligned}$$

which completes the proof in the case $p \in [1, \infty)$.

In the case $p = \infty$, we choose an arbitrary $\omega \Subset \Omega$. Then $u \in W^{1,p}(\omega)$, for all $p \in [1, \infty)$, and so the equality in (6.1.11) holds in ω, so in Ω. $\square$

We note that Proposition 6.1.12 does not hold in general if h is not C^1. For example, let $\Omega = (0, 1)$, $u = x^\alpha$, $\alpha \in (1/2, 1/\sqrt{2})$, and $h(x) = x^\alpha$. From Example 6.1.3, we have $u, h \in H^1(\Omega)$ because $p(\alpha - 1) > -N$ (here $p = 2$ and $N = 1$), but $h \circ u(x) = x^{\alpha^2} \notin H^1(\Omega)$ because $p(\alpha^2 - 1) < -N$.

Proposition 6.1.13 *Let $\Omega, G \subset \mathbb{R}^N$ be open sets and $\theta = (\theta_1, \ldots, \theta_N) \in W^{1,\infty}(G; \Omega)$ be invertible with $\theta^{-1} \in W^{1,\infty}(\Omega; G)$. If $u \in W^{1,p}(\Omega)$, $p \in [1, \infty]$, then $u \circ \theta \in W^{1,p}(G)$ and*

$$D^\alpha(u \circ \theta) = \sum_{|\beta|=1} D^\alpha \theta^\beta D^\beta u \circ \theta, \quad \forall \alpha \in \mathbb{N}_0^N, \; |\alpha| = 1, \tag{6.1.12}$$

where $\theta^\beta = \theta_1^{\beta_1} \cdots \theta_N^{\beta_N}$.

Proof. Let $u \in W^{1,p}(\Omega)$. From the classical change of variable theorem for $L^p(\Omega)$ functions, we have

$$\forall u \in L^p(\Omega), \quad u \circ \theta \in L^p(G) \quad \textit{and} \quad \int_\Omega |u|^p dx = \int_G |u \circ \theta|^p |D\theta| dy. \tag{6.1.13}$$

So $u \circ \theta$ and all the right-hand side terms of (6.1.12) are in $L^p(G)$. Then to prove $u \circ \theta \in W^{1,p}(G)$, it is enough to prove (6.1.12).

Consider first the case $p \in [1,\infty)$. For $\varphi \in \mathcal{D}(\Omega)$, there exists $\omega \Subset \Omega$ such that $\mathrm{supp}(\varphi) \subset \omega$. Let (u_n) in $\mathcal{D}(\Omega)$, $\lim_{n\to\infty} \|u_n - u\|_{W^{1,p}(\omega)} = 0$, as in Theorem 6.1.8. Note that (6.1.13) implies that $\lim\limits_{n\to\infty} u_n \circ \theta = u \circ \theta$ and $\lim\limits_{n\to\infty} D^\alpha u_n \circ \theta = D^\alpha u \circ \theta$ in $L^p(G)$, for all $|\alpha| = 1$. Therefore we get

$$\begin{aligned}
\langle D^\alpha(u\circ\theta), \varphi\rangle &= -\langle u\circ\theta, D^\alpha\varphi\rangle = -\lim_{n\to\infty}\langle u_n\circ\theta, D^\alpha\varphi\rangle \\
&= \lim_{n\to\infty}\langle D^\alpha(u_n\circ\theta), \varphi\rangle \\
&= \lim_{n\to\infty}\sum_{|\beta|=1}\langle D^\alpha\theta^\beta D^\beta u_n\circ\theta, \varphi\rangle \\
&= \sum_{|\beta|=1}\langle D^\alpha\theta^\beta D^\beta u\circ\theta, \varphi\rangle,
\end{aligned}$$

which proves (6.1.12) and $u\circ\theta \in W^{1,p}(G)$.

In the case $p = \infty$, we can choose an open and bounded set ω as in Proposition 6.1.12. Then $u \in W^{1,p}(\omega)$, for all $p \in [1,\infty)$, and (6.1.12) holds in $\theta^{-1}(\omega)$, which from the choice of ω proves (6.1.12). $\square$

Note that Proposition 6.1.13 extends in the $W^{k,p}$ context. Namely, let $\Omega, G \subset \mathbb{R}^N$ be two open sets and $\theta = (\theta_1, \ldots, \theta_N) \in W^{k,\infty}(G;\Omega)$ be invertible with $\theta^{-1} \in W^{k,\infty}(\Omega;G)$. If $u \in W^{k,p}(\Omega)$ then $u\circ\theta \in W^{k,p}(G)$, $p \in [1,\infty]$, and, furthermore, $D^\alpha(u\circ\theta)$, $|\alpha| = k$, is a finite sum of terms of the form $D^\beta u\circ\theta$, $|\beta| \le k$, multiplied by $D^\gamma\theta^\sigma$ terms, with $|\gamma| \le k$, $|\sigma| = 1$.

The proof is carried out by recurrence on k. Indeed, for $k = 1$ the claim holds. Assume it holds for $k-1$, so $D^\alpha(u\circ\theta)$, $|\alpha| = k-1$, is a sum of terms $D^\beta u\circ\theta$, $|\beta| \le k-1$, multiplied by $D^\gamma\theta^\sigma$ terms, with $|\gamma| \le k-1$, $|\sigma| = 1$. Then applying Proposition 6.1.13 to $D^\alpha(u\circ\theta)$, $|\alpha| = k-1$, proves the claim.

Example 6.1.14 *As a corollary of Proposition 6.1.12, let us prove the following. Let $\Omega \subset \mathbb{R}^N$ be an open set, $u \in W^{1,p}(\Omega)$, $p \in [1,\infty]$, and set*

$$u^+ = \max\{0,\ u(x),\ x \in \Omega\}, \quad u^- = \min\{0,\ u(x),\ x \in \Omega\}. \tag{6.1.14}$$

Then

$$u^\pm \in W^{1,p}(\Omega), \tag{6.1.15}$$
$$\nabla u^\pm = (\nabla u)\mathbb{1}_{\{u \gtrless 0\}}, \quad \textit{and} \tag{6.1.16}$$
$$\nabla u = 0 \ \textit{ a.e. in } \{u = C\} := \{x \in \Omega,\ u(x) = C\},\ \ C \in \mathbb{R}. \tag{6.1.17}$$

Indeed, for $\epsilon > 0$ let $f_\epsilon(t) = ((t^2+\epsilon^2)^{1/2} - \epsilon)H(t)$, and $u_\epsilon = f_\epsilon \circ u$. From Proposition 6.1.12, we have $u_\epsilon \in W^{1,p}(\Omega)$ and for $\varphi \in \mathcal{D}(\Omega)$ we get

$$\langle Du^+, \varphi\rangle = -\int_\Omega u^+ D\varphi dx = -\lim_{\epsilon\to 0}\int_\Omega u_\epsilon D\varphi dx$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\int_\Omega Du_\epsilon \varphi dx = \lim_{\epsilon\to 0}\int_{\{u>0\}} \frac{u\nabla u}{(u^2+\epsilon^2)^{1/2}}\varphi \\
&= \int_\Omega \mathbb{1}_{\{u>0\}}\nabla u\varphi,
\end{aligned}
$$

which proves the lemma for u^+. The result for u^- follows from the result for u^+ by noting that $u^- = -(-u)^+$. For (6.1.17) set $v = u - C$. Since $u = C + v^+ + v^-$, from (6.1.16) it follows

$$
(\nabla u)\mathbb{1}_{\{u=C\}} = (\nabla v)\mathbb{1}_{\{v=0\}} = (\nabla v^+ + \nabla v^-)\mathbb{1}_{\{v=0\}} = 0.
$$

6.2 H^s spaces and Fourier transform: $W^{s,p}$ and $W_0^{s,p}$ spaces

Sobolev spaces in $\mathbb{R}^N$ are easily analyzed by using Fourier transform. The advantage of working with these spaces is that they are homeomorphic to the space L^2_s defined below, which makes it possible to "*transfer*" well-known properties of L^2_s spaces to H^s spaces.

Theorem 6.2.1 *For $u \in H^k$ and $\alpha \in \mathbb{N}_0^N$, $|\alpha| \le k$, we have*

$$
\mathcal{F}[D^\alpha u] = (i\xi)^\alpha \mathcal{F}[u], \quad \|D^\alpha u\|_{L^2} = \|(i\xi)^\alpha \mathcal{F}[u]\|_{L^2}. \tag{6.2.1}
$$

Furthermore, Fourier transform $\mathcal{F}$ is a homeomorphism[1] *from H^k onto the weighted space L^2_k defined by*

$$
L^2_k = \left\{ v \in L^2,\ \|v\|^2_{L^2_k} := \int_{\mathbb{R}^N} \langle\xi\rangle^{2k}|v(\xi)|^2 d\xi < \infty, \quad \langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}} \right\}. \tag{6.2.2}
$$

Proof. Note that from Theorem 5.4.9 and Proposition 5.4.6, (6.2.1) holds in $\mathcal{S}$.
i) $\mathcal{F} : H^k \mapsto L_k$ is well-defined and continuous. Indeed, for $u \in \mathcal{S}$, from (6.2.1) we get

$$
\begin{aligned}
\|\mathcal{F}[u]\|^2_{L^2_k} &= \int_{\mathbb{R}^N} \langle\xi\rangle^{2k}|\hat{u}|^2 d\xi \sim \int_{\mathbb{R}^N} \sum_{|\alpha|\le k} |(i\xi)^\alpha|^2 |\hat{u}|^2 d\xi = \sum_{|\alpha|\le k} \|\mathcal{F}[D^\alpha u]\|^2_{L^2(\mathbb{R}^N)} \\
&= \|u\|^2_{H^k}.
\end{aligned} \tag{6.2.3}
$$

As $\mathcal{S} \xhookrightarrow{d} H^k$, (6.2.3) implies that $\mathcal{F}$ extends to a continuous linear operator from H^k to L^2_k and (6.2.1) holds for all $u \in H^k$.
ii) $\mathcal{F} : H^k \mapsto L^2_k$ is invertible and its inverse $\mathcal{F}^{-1}$ is continuous. Indeed, let $v \in L^2_k$. Then $v \in L^2$ and from Theorem 5.4.9 there exists a unique $u \in L^2$ such that $\mathcal{F}[u] = v$. From Remark 5.4.11 and $v \in L^2_k$, it follows $\mathcal{F}[D^\alpha u] = (i\xi)^\alpha v \in L^2$ for all $|\alpha| \le k$, which in

[1] A continuous linear invertible operator from a normed space X onto another normed space Y.

combination with Theorem 5.4.9 implies $D^\alpha u \in L^2$, so $u \in H^k$ which proves that $\mathcal{F}$ is invertible. Finally, the boundedness of $\mathcal{F}^{-1}$ follows from (6.2.3). □

The estimate (6.2.3) shows that $\|\hat{u}\|_{L^2_k}$ is a norm in H^k, equivalent to the usual norm $\|u\|^2_{H^k} = \sum_{|\alpha|\le k} \|D^\alpha u\|^2_{L^2}$, which motivates the introduction of the spaces H^s, with arbitrary $s > 0$.

Definition 6.2.2 *For $s > 0$, the "fractional order Sobolev space H^s" is defined by*

$$
\begin{aligned}
H^s &= \{u \in L^2, \quad \hat{u} := \mathcal{F}[u] \in L^2_s\}, \quad \textit{equipped with} && (6.2.4)\\
(u,v)_{H^s} &= (\hat{u},\hat{v})_{L^2_s}, && (6.2.5)
\end{aligned}
$$

where

$$
\begin{aligned}
L^2_s &= \left\{ v \in L^2, \ \|v\|^2_{L^2_s} = (v,v)_{L^2_s} < \infty \right\}, \quad \textit{and} && (6.2.6)\\
(v,w)_{L^2_s} &= \int_{\mathbb{R}^N} (\langle\xi\rangle^s v)(\overline{\langle\xi\rangle^s w})d\xi. && (6.2.7)
\end{aligned}
$$

Note that in some texts is used $\langle\xi\rangle^{2s} = 1+|\xi|^{2s}$. The results are exactly the same because $(1+|\xi|^2)^s$ and $1+|\xi|^{2s}$ are equivalent for $s > 0$. This is the reason why we will use $\langle\xi\rangle^{2s} = 1+|\xi|^{2s}$, whenever it is more appropriate.

Example 6.2.3 *Let $u(x) = \mathbb{1}_{(0,1)}(x)$, $x \in \mathbb{R}$. Clearly $u \in L^2$. As $u(x) = H(x) - H(x-1)$, we get $u' = \delta_0 - \delta_1$. So $u' \notin L^2$ and therefore $u \notin H^1$. But, for what $s > 0$ do we have $u \in H^s$? Note that we have*

$$
\begin{aligned}
\hat{u} &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(\xi x)} u(x)dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i(\xi x)}dx = \frac{1}{\sqrt{2\pi}} \frac{1-e^{-i\xi}}{i\xi},\\
\int_{\mathbb{R}} |\xi|^{2s}|\hat{u}|^2 d\xi &= C\int_{\mathbb{R}} |\xi|^{2(s-1)}|1-\cos(\xi)|^2 d\xi\\
&< \infty \quad iff \quad 2(s-1) < -1 \quad \textit{or equivalently} \quad s < \frac{1}{2}.
\end{aligned}
$$

Hence, $\mathbb{1}_{(0,1)} \in H^s$ only for $s \in (0, 1/2)$.

Example 6.2.4 *Let $u \in H^s$, $s > 0$, and $\alpha \in \mathbb{N}_0^N$, $|\alpha| \le s$. Then $D^\alpha u \in H^{s-|\alpha|}$. Indeed,*

$$
\begin{aligned}
\mathcal{F}[D^\alpha u] &= (i\xi)^\alpha \mathcal{F}[u],\\
\int_{\mathbb{R}^N} \langle\xi\rangle^{2(s-|\alpha|)} |\mathcal{F}[D^\alpha u]|^2 d\xi &= \int_{\mathbb{R}^N} \langle\xi\rangle^{2(s-|\alpha|)} |(i\xi)^\alpha|^2 |\mathcal{F}[u]|^2 d\xi\\
&\le C\int_{\mathbb{R}^N} \langle\xi\rangle^{2(s-|\alpha|)+2|\alpha|} |\mathcal{F}[u]|^2 d\xi\\
&= C\|\mathcal{F}[u]\|^2_{L^2_s}\\
&< \infty,
\end{aligned}
$$

which proves the claim.

Remark 6.2.5 *There is another norm for H^s spaces which serves as motivation for defining $W^{s,p}(\Omega)$ spaces. Indeed, let $s \in (0,1)$ and $u \in H^s$. Using the fact*

$$\begin{aligned}\mathcal{F}[u(\cdot+y)](\xi) &= \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-i(\xi\cdot x)}u(x+y)dx = e^{i(\xi\cdot y)}\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-i(\xi\cdot z)}u(z)dz \\ &= e^{i(\xi\cdot y)}\mathcal{F}[u](\xi),\end{aligned}$$

and Plancherel theorem, one gets

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x+y)-u(x)|^2}{|y|^{N+2s}}dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\mathcal{F}[u(\cdot+y)](\xi)-\mathcal{F}[u](\xi)|^2}{|y|^{N+2s}}d\xi dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e^{i(y\cdot\xi)}-1|^2}{|y|^{2s}}\cdot\frac{1}{|y|^N}\cdot|\mathcal{F}[u](\xi)|^2 d\xi dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e^{i(y\cdot\xi)}-1|^2}{|y|^{2s}}\cdot\frac{dy}{|y|^N}|\mathcal{F}[u](\xi)|^2 d\xi \\ &=: \int_{\mathbb{R}^N} h(\xi)|\mathcal{F}[u](\xi)|^2 d\xi.\end{aligned}$$

Note that $h(\xi)$ is radial and homogeneous of degree $2s$:

$$h(t\xi) = \int_{\mathbb{R}^N}\frac{|e^{i(y\cdot(t\xi))}-1|^2}{|y|^{2s}}\cdot\frac{dy}{|y|^N} = \int_{\mathbb{R}^N}\frac{|e^{i(z\cdot\xi)}-1|^2}{|z|^{2s}|t|^{-2s}}\cdot\frac{|t|^{-N}}{|t|^{-N}}\cdot\frac{dz}{|z|^N} = |t|^{2s}h(\xi),$$

which implies[2]

$$h(\xi) = h\left(|\xi|\frac{\xi}{|\xi|}\right) = |\xi|^{2s}h\left(\frac{\xi}{|\xi|}\right) = C_s|\xi|^{2s}, \qquad C_s = \int_{\mathbb{R}^N}\frac{|e^{i(y\cdot\zeta)}-1|^2}{|y|^{2s}}\cdot\frac{dy}{|y|^N}, \quad |\zeta| = 1.$$

Therefore, for $s \in (0,1)$ and $u \in H^s$ we get

$$C_s\int_{\mathbb{R}^N}|\xi|^{2s}|\mathcal{F}[u]|^2 d\xi = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy. \tag{6.2.8}$$

Note that for $s \in \mathbb{R}_+$, we have the equivalence

$$\langle\xi\rangle^{2s} \sim \langle\xi\rangle^{2\lfloor s\rfloor} + |\xi|^{2(s-\lfloor s\rfloor)}\sum_{|\alpha|=\lfloor s\rfloor}|\xi^\alpha|^2, \tag{6.2.9}$$

where $\lfloor s\rfloor$ is the largest integer smaller than s. Hence from (6.2.8), for $0 < s \notin \mathbb{N}$ we get

$$\|u\|_{H^s}^2 \sim \int_{\mathbb{R}^N}\langle\xi\rangle^{2\lfloor s\rfloor}|\mathcal{F}[u]|^2 d\xi + \sum_{|\alpha|=\lfloor s\rfloor}\int_{\mathbb{R}^N}|\xi|^{2(s-\lfloor s\rfloor)}|(i\xi)^\alpha\mathcal{F}[u]|^2 d\xi$$

[2] Here $C_s < \infty$ iff $s \in (0,1)$.

$$= \|u\|^2_{H^{\lfloor s\rfloor}} + \sum_{|\alpha|=\lfloor s\rfloor} \int_{\mathbb{R}^N} |\xi|^{2(s-\lfloor s\rfloor)} |\mathcal{F}[D^\alpha u]|^2 d\xi$$
$$= \|u\|^2_{H^{\lfloor s\rfloor}} + \sum_{|\alpha|=\lfloor s\rfloor} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{N+2(s-\lfloor s\rfloor)}} dxdy.$$

This implies that for $s \in \mathbb{R}_+ \backslash \mathbb{N}$, an equivalent inner product in H^s is given by

$$\begin{aligned} (u,v)_{H^s} &= (u,v)_{H^{\lfloor s\rfloor}} \\ &+ \sum_{|\alpha|=\lfloor s\rfloor} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha \overline{v}(x) - D^\alpha \overline{v}(y))}{|x-y|^{N+2(s-\lfloor s\rfloor)}} dxdy, \end{aligned} \tag{6.2.10}$$

which motivates the following definition.

Definition 6.2.6 *Let $\Omega \subseteq \mathbb{R}^N$ open.*

i) For $s \in \mathbb{R}_+ \backslash \mathbb{N}$, $p \in [1,\infty)$, define $(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}(\Omega)})$ by

$$\begin{aligned} W^{s,p}(\Omega) &= \left\{ u \in L^p(\Omega), \ \ \|u\|^p_{W^{s,p}(\Omega)} := \|u\|^p_{W^{\lfloor s\rfloor,p}(\Omega)} + |u|^p_{W^{s,p}(\Omega)} < \infty \right\}, \quad (6.2.11) \\ |u|^p_{W^{s,p}(\Omega)} &= \sum_{|\alpha|=\lfloor s\rfloor} \left\| \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^{\frac{N}{p}+(s-\lfloor s\rfloor)}} \right\|^p_{L^p(\Omega\times\Omega)}, \end{aligned}$$

where $\|\cdot\|_{W^{\lfloor s\rfloor}(\Omega)}$ is defined in (6.1.2).

ii) For $s > 0$ and $p \in [1,\infty)$, we denote by $W_0^{s,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ for the norm (6.2.11) or (6.1.2).

iii) If $p = 2$ we write $H^s(\Omega) = W^{s,2}(\Omega)$, $H_0^s(\Omega) = W_0^{s,2}(\Omega)$, and equip them with the inner product

$$\begin{aligned} (u,v)_{H^s(\Omega)} &:= (u,v)_{H^{\lfloor s\rfloor,p}(\Omega)} \\ &+ \sum_{|\alpha|=\lfloor s\rfloor} \int_\Omega \int_\Omega \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha \overline{v}(x) - D^\alpha \overline{v}(y))}{|x-y|^{N+2(s-\lfloor s\rfloor)}} dxdy. \end{aligned} \tag{6.2.12}$$

iv) The number $s - \frac{N}{p}$ is called a "differential dimension of $W^{s,p}(\Omega)$".

Note that the spaces $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ are Banach spaces and for $p = 2$, they are Hilbert spaces; see, for example, [1, 44, 51]. Also, it follows H^s of (6.2.10) and is the same as $W^{s,2}$ of (6.2.11).

Remark 6.2.7 *The case $s \notin \mathbb{N}$ and $p = \infty$ is excluded from Definition 6.2.6. The reason is that $W^{s,\infty}(\Omega) = C_b^{\lfloor s \rfloor, s-\lfloor s \rfloor}(\Omega)$; see Proposition 1.1.11. Indeed, by formally letting $p \to \infty$ in Definition 6.2.6, one would define (see, for example, [41])*

$$W^{s,\infty}(\Omega) = \Bigg\{ u \in L^\infty(\Omega) \quad \textit{such that} \qquad \|u\|_{W^{s,\infty}(\Omega)} := \|u\|_{W^{\lfloor s \rfloor,\infty}(\Omega)} + \sum_{|\alpha|=\lfloor s \rfloor} \left\| \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^{s-\lfloor s \rfloor}} \right\|_{L^\infty(\Omega\times\Omega)} < \infty \Bigg\}. \tag{6.2.13}$$

Example 6.2.8 *Let $\Omega = (-1,1) \subset \mathbb{R}$ and $u(x) = H(x)$, $x \in \Omega$. We wonder for what (s,p) we have $u \in W^{s,p}(\Omega)$. We know that $u \notin W^{1,p}(\Omega)$, for all $p \in [1,\infty]$. Then we restrict the discussion to the case $s \in (0,1)$, $p \in [1,\infty)$. Clearly $u \in L^p(\Omega)$. We compute*

$$\begin{aligned}
|u|^p_{W^{s,p}(\Omega)} &= \int_{-1}^1 \int_{-1}^1 \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} dxdy = 2 \int_{-1}^0 \int_0^1 \frac{1}{(x-y)^{1+sp}} dxdy \\
&= -\frac{2}{sp} \int_{-1}^0 (x-y)^{-sp}\Big|_{x=0}^{x=1} dy = -\frac{2}{sp} \int_{-1}^0 \Big((1-y)^{-sp} - (0-y)^{-sp} \Big) dy \\
&= \frac{2}{sp} \frac{1}{sp-1} \Big[((1-y)^{-sp+1} - (0-y)^{-sp+1} \Big]_{-1}^0 \\
&< \infty, \quad iff \quad sp < 1.
\end{aligned}$$

So, $u \in W^{s,p}(\Omega)$ for all $s \in (0,1)$, $p \in [1,\infty)$ such that $sp < 1$.

6.3 Continuous, compact, and dense embedding theorems in $H^s(\Omega)$

Continuous and compact embedding results hold in $W^{s,p}$ spaces. In order to simplify the proofs, we will restrict our discussion to H^s spaces. Let us start with a general definition.

Definition 6.3.1 *Let E and F be two Banach spaces.*

i) We say "E is continuously embedded in F", and write "$E \hookrightarrow F$", if[3]

$$E \subset F \ \ \textit{and} \ \ \exists C > 0, \ \forall u \in E, \quad \|u\|_F \leq C\|u\|_E. \tag{6.3.1}$$

ii) We say "E is compactly embedded in F", and we write "$E \overset{c}{\hookrightarrow} F$", if $E \hookrightarrow F$ and the unit ball in E is precompact in F, i.e. every bounded sequence in E has a convergent subsequence in F.

[3] In other words, the map $I : u \in E \mapsto I(u) = u \in F$ is continuous.

iii) We say "$\mathcal{H}$ is dense in E", or "$\mathcal{H}$ is densely embedded in E", and write "$\mathcal{H} \overset{d}{\hookrightarrow} E$", if $\mathcal{H} \subset E$ and for every element $u \in E$ there exists a sequence (u_n) in $\mathcal{H}$ such that $\lim\limits_{n\to\infty} u_n = u$ in E.

Continuous, resp. compact, embedding theorems in Sobolev spaces show that if the differential dimension $s - \frac{N}{p}$ is large enough then the Sobolev space $W^{s,p}$ is continuously embedded in, for example, more familiar spaces such as C^0, resp. compactly embedded in larger spaces, such as L^2. Compact embeddings are particularly important when studying nonlinear PDEs.

Density theorems are significant when $\mathcal{H}$ is a set of regular functions. For example, one can prove a certain property only for $u \in \mathcal{H}$, which a priori is simpler than the proof for $u \in E$. Next, by using the density of $\mathcal{H}$ in E, one can prove that the same property holds for $u \in E$. This is done by writing the property for a sequence (u_n) in $\mathcal{H}$, with $\lim_{n\to\infty} u_n = u$ in E, and then passing to the limit.

6.3.1 Case $\Omega = \mathbb{R}^N$

First, we develop the results in H^s spaces. Here, the analysis is simplified based on the fact that H^s spaces are homeomorphic with L^2_s spaces. Next, the results are extended to $H^s(\Omega)$ spaces. Here, the regularity of Ω is important. It allows us to extend $H^s(\Omega)$ functions as H^s functions and use the results developed for H^s spaces.

Theorem 6.3.2 *For all $s > 0$, the space H^s equipped with the inner product $(u, v)_{H^s}$ is a Hilbert space. Furthermore, $\mathcal{D} \overset{d}{\hookrightarrow} H^s$ and $\mathcal{S} \overset{d}{\hookrightarrow} H^s$.*

Proof. From Definition 6.2.2, $\mathcal{F} : H^s \mapsto L^2_s$ is an isometry,[4] which implies that $(H^s, (u, v)_{H^s})$ is an Hilbert space because $(L^2_s, \|\cdot\|_{L^2_s})$ is so. The embedding $\mathcal{S} \overset{d}{\hookrightarrow} H^s$ follows from $\mathcal{S} \overset{d}{\hookrightarrow} L^2_s$, $\mathcal{F}[\mathcal{S}] = \mathcal{S}$ and the isometry of $\mathcal{F} : H^s \mapsto L^2_s$.

For the embedding $\mathcal{D} \overset{d}{\hookrightarrow} H^s$, given $u \in H^s$, let (u_n) in $\mathcal{S}$ converging to u in H^s. As $\mathcal{D} \overset{d}{\hookrightarrow} \mathcal{S}$, see Proposition 5.4.4, for every u_n there exists (φ_m) in $\mathcal{D}$ such that (φ_m) converges to u_n in $\mathcal{S}$. From the continuity of $\mathcal{F} : \mathcal{S} \mapsto \mathcal{S}$ in $\mathcal{S}$, we get

$$\begin{aligned}
\|u_n - \varphi_m\|^2_{H^s(\mathbb{R}^N)} &= \|\hat{u}_n - \hat{\varphi}_m\|^2_{L^2_s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \langle\xi\rangle^{2s} |\hat{u}_n - \hat{\varphi}_m|^2 d\xi \\
&\leq \int_{\mathbb{R}^N} \frac{1}{\langle\xi\rangle^{2N}} \langle\xi\rangle^{2(N+s)} |\hat{u}_n - \hat{\varphi}_m|^2 d\xi \\
&\leq C_N |s_{(N+\lfloor s\rfloor)}(\hat{u}_n - \hat{\varphi}_m)|^2 \\
&\xrightarrow[m\to\infty]{} 0.
\end{aligned}$$

We take $\psi_n = \varphi_{m_n}$ with m_n such that $\|u_n - \varphi_{m_n}\|_{H^s(\mathbb{R}^N)} < 1/n$. So (ψ_n) is in $\mathcal{D}$ and converges to u in H^s.

[4] A homeomorphism that preserves the norm.

Theorem 6.3.3 *Let $s - \frac{N}{2} > 0$. Then*[5] $H^s \longrightarrow C_b^0 \cap \{u,\ \lim_{|x|\to\infty} u(x) = 0\}$.

Proof. Note that $u(x) = \mathcal{F}^{-1}[\mathcal{F}[u](\xi)](x)$, so $u(-x) = \mathcal{F}[\mathcal{F}[\xi]](x)$. Recall also Riemann-Lebesgue Lemma 9.4.14, which says if $v \in L^1(\mathbb{R}^N)$ and $\hat{v} = \mathcal{F}[v]$ then

$$\hat{v} \in C_b^0 \cap \{\lim_{|\xi|\to\infty} \hat{v}(\xi) = 0\} \ \ \textit{and} \ \ \|\hat{v}\|_{L^\infty} \leq \frac{1}{(2\pi)^{N/2}}\|v\|_{L^1}.$$

If $\hat{u} \in L^1$, applying this lemma with $v = \hat{u}$ gives the required result. So, we need to show that $\hat{u} \in L^1$. As $u \in H^s$ we have $\hat{u} \in L_s^2$ and

$$\int_{\mathbb{R}^N} |\hat{u}| d\xi \ = \ \|\langle\xi\rangle^{-s}\langle\xi\rangle^s\hat{u}\|_{L^1} \leq \|\langle\xi\rangle^{-s}\|_{L^2}\|\langle\xi\rangle^s\hat{u}\|_{L^2} \leq C\left(\int_{\mathbb{R}^N}\frac{d\xi}{\langle\xi\rangle^{2s}}\right)^{1/2}\|u\|_{H^s}.$$

Since $s - \dfrac{N}{2} > 0$, we obtain

$$\int_{\mathbb{R}^N}\frac{d\xi}{\langle\xi\rangle^{2s}} \ = \ \int_1^\infty \frac{r^{N-1}}{(1+|r|^2)^s}dr \sim \left(1 + \int_1^\infty r^{N-1-2s}ds\right) < \infty,$$

which proves $\|\hat{u}\|_{L^1} < \infty$ and completes the proof. $\square$

For the proof of the following result, see Theorem 9.5.1 in Annex.

Theorem 6.3.4 *Assume $s - \frac{N}{2} \in (m, m+1]$, $m \in \mathbb{N}_0$ and let $\sigma \in (0, s - \frac{N}{2} - m\})$. Then*

$$H^s \longrightarrow C_b^{m,\sigma} \cap \{u,\ \lim_{|x|\to\infty} D^\alpha u = 0,\ \forall\alpha \in \mathbb{N}_0^N,\ |\alpha| \leq m\}. \tag{6.3.2}$$

In particular, there exists $C > 0$ such that for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = m$ and $x, y \in \mathbb{R}^N$ we have

$$|D^\alpha u(x) - D^\alpha u(y)| \leq C\|D^\alpha u\|_{H^{s-m}}|x-y|^\sigma. \tag{6.3.3}$$

Remark 6.3.5 *One may wonder whether the condition $s - \frac{N}{2} > 0$ in theorems 6.3.3 and 6.3.4 is due to the technique we have used. Actually, this condition is necessary, in the sense that if $s - \frac{N}{2} \leq 0$ the theorem does not hold. Indeed, we have seen in Example 6.2.3 that $u = \mathbb{1}_{(0,1)} \in H^s(\mathbb{R})$ only for $s - 1/2 < 0$ (so, here $N = 1$ and $s - N/2 = s - 1/2 < 0$). Clearly $u \notin C^0$. In this context, we say that the condition $s - \frac{N}{2} > 0$ in theorems 6.3.3 and 6.3.4 is optimal.*

Theorems 6.3.3 and 6.3.4 do not hold in general even in the critical case $s - N/2 = 0$. For example, let $u(x) = \mathbb{1}_{\{|x|<1\}} \ln|1 - \ln|x||$, $x \in \mathbb{R}^2$. Note that $u \in H^1(\mathbb{R}^2)$, however $u \notin C^0$ (here, $s - N/2 = 1 - 2/2 = 0$). The proof of $u \in H^1(\mathbb{R}^2)$ goes as follows:

$$\int_{\mathbb{R}^2} |u|^2 dx \ = \ \int_{\{|x|<1\}} |u|^2 dx \leq C\int_0^1 r|\ln|1 - \ln r|^2 dr < \infty,$$

[5]We remind that if a space symbol is not followed by a domain, then it is understood that the domain is $\mathbb{R}^N$.

and as $\partial_i u = \mathbb{1}_{\{|x|<1\}}\partial_i \ln|1-\ln|x||$ *we get*

$$\begin{aligned}\int_{\mathbb{R}^2} |\partial_i u|^2 dx &= \int_{\{|x|<1\}} \frac{1}{|1-\ln|x||^2}\cdot\frac{1}{|x|^2}\cdot\frac{|x_i|^2}{|x|^2}dx \le \int_{\{|x|<1\}} \frac{1}{|1-\ln|x||^2}\cdot\frac{dx}{|x|^2}\\ &\le C\int_0^1 \frac{1}{|1-\ln r|^2}\cdot\frac{dr}{r} = C\int_0^1 ((1-\ln r|)^{-1})' dr\\ &< \infty, \quad i=1,2.\end{aligned}$$

6.3.2 Case $\Omega \subsetneq \mathbb{R}^N$

We start with the definition of "*extension*" and "*restriction*" operators. They are used to extend $H^s(\Omega)$ in H^s and then to restrict H^s to $H^s(\Omega)$, which makes it possible to apply H^s embedding results to $H^s(\Omega)$ case.

Definition 6.3.6 *Let $\Omega \subsetneq \mathbb{R}^N$ be an open set, $s > 0$ and $p \in [1,\infty]$.*

i) A linear continuous operator E (or E_Ω), $E : W^{s,p}(\Omega) \mapsto W^{s,p}$, satisfying $U = Eu$, $U(x) = u(x)$ for all $x \in \Omega$, is called a "$W^{s,p}(\Omega)$ extension operator".

We say that a domain Ω satisfies "$W^{s,p}$ extension property" if there exists a $W^{s,p}(\Omega)$ extension operator. If $p = 2$, we say that E is an "$H^s(\Omega)$ extension operator" and Ω satisfies the "H^s extension property".

ii) A linear continuous operator R (or R_Ω), $R : W^{s,p} \mapsto W^{s,p}(\Omega)$, satisfying $u = RU$, $u(x) = U(x)$ for all $x \in \Omega$, is called a "$W^{s,p}(\Omega)$ restriction operator".

The existence of the restriction operator R is straightforward from Definition 6.2.6, which shows that R is well-defined, linear, and continuous. The existence of the extension operator is a difficult problem that we will consider below. Note also that whenever E exists, it is a right inverse of R.

Theorem 6.3.7 *If Ω is a bounded Lipschitz domain, then it satisfies $W^{s,p}$ extension property for all $s > 0$ and $p \in [1,\infty]$.*

For the proof of the theorem in the case $s \in \mathbb{N}$ and $p \in [1,\infty]$, see [48]. For the proof in the case $0 < s \notin \mathbb{N}$ and $p \in (1,\infty)$ and a proof by interpolation, see [54]. For a general direct proof, see [45].

The theorem below gives a detailed instructive step-by-step proof in the case $s = 1$, $p \in [1,\infty]$, and $\Omega = \mathbb{R}^N_+$ of the existence of an extension operator.

Theorem 6.3.8 *There exists a linear continuous extension operator $E : W^{1,p}(\mathbb{R}^N_+) \mapsto W^{1,p}(\mathbb{R}^N)$, $p \in [1,\infty]$.*

Proof. For $u \in W^{1,p}(\mathbb{R}^N_+)$, let $U = Eu$ be given by

$$U(x', x_N) = \begin{cases} u(x', x_N), & x_N \geq 0, \\ u(x', -x_N), & x_N < 0. \end{cases} \tag{6.3.4}$$

E is the required extension operator. Indeed, clearly E is linear. Let us prove that E is well-defined and continuous.
(i) E is well-defined, so if $u \in W^{1,p}(\mathbb{R}^N_+)$ then $U \in W^{1,p}(\mathbb{R}^N)$.
(i.1) Let $\alpha = (\alpha', \alpha_N) \in \mathbb{N}_0^N$, $|\alpha| = 1$. Then for $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we have

$$\begin{aligned} \langle D^\alpha U, \varphi \rangle &= -\int_{\mathbb{R}^N} U(x) D^\alpha \varphi(x) dx \\ &= -\left(\int_{\mathbb{R}^N_+} u(x) D^\alpha \varphi(x) dx + \int_{\mathbb{R}^N_-} u(x', -x_N) D^\alpha \varphi(x', x_N) dx \right) \\ &= -\int_{\mathbb{R}^N_+} u(x) D^\alpha \psi(x) dx, \end{aligned} \tag{6.3.5}$$

with

$$\psi(x', x_N) = \varphi(x', x_N) + (-1)^{\alpha_N} \varphi(x', -x_N),$$

which we found after using the change of variable formulas (see Proposition 6.1.13)

$$\int_{\mathbb{R}^N_\pm} u(x', -x_N) D^\alpha_x v(x', x_N) dx = (-1)^{\alpha_N} \int_{\mathbb{R}^N_\mp} u(y', y_N) D^\alpha_y (v(y', -y_N)) dy. \tag{6.3.6}$$

(i.2) Note that in general $\psi \notin \mathcal{D}(\mathbb{R}^N_+)$ and therefore, a priori, we cannot pass D^α on u in (6.3.5). However, ψ is "*small*" on $\partial\mathbb{R}^N_+$ and this is sufficient to pass D^α on u. We proceed as follows. Let $\eta \in C^\infty(\mathbb{R})$, $\eta = 0$ in $(-\infty, 1/2]$, $\eta = 1$ in $[1, \infty)$ and set $\eta_n(x_N) = \eta(n x_N)$. Note that

$$\begin{aligned} \int_{\mathbb{R}^N_+} u D^\alpha \psi dx &= \lim_{n \to \infty} \int_{\mathbb{R}^N_+} u(\eta_n D^\alpha \psi) dx \\ &= \lim_{n \to \infty} \int_{\mathbb{R}^N_+} \Big(u D^\alpha (\eta_n \psi) - u \psi D^\alpha \eta_n \Big) dx. \end{aligned} \tag{6.3.7}$$

We claim that the limit of the second term in (6.3.7) is zero. Indeed, if $\alpha_N = 0$ then $D^\alpha \eta_n = 0$. If $\alpha_N = 1$ we get

$$D^\alpha \eta_n(x_N) = n\eta'(n x_N), \quad \psi(x', 0) = 0, \quad \psi(x', x_N) = x_N D^\alpha \psi(x', \theta x_N),$$

where $\theta = \theta(x) \in (0, 1)$, which implies

$$\left| \int_{\mathbb{R}^N_+} u \psi D^\alpha \eta_n dx \right| = \left| \int_{\mathbb{R}^N_+} u(x)(x_N D^\alpha \psi(\theta x))(n\eta'(n x_N)) dx \right|$$

$$\begin{aligned} &\leq \quad \|\eta\|_{C^1(\mathbb{R})}\|\psi\|_{C^1(\mathbb{R}^N)} \int_{\{|x_N|\leq 1/n\}} |u|dx \\ &\xrightarrow[n\to\infty]{} \quad 0, \end{aligned} \tag{6.3.8}$$

where we used Lebesgue DCT and the fact that $\eta'(nx_N) = 0$ for $x_n > \frac{1}{n}$.

Therefore as $\eta_n\psi \in \mathscr{D}(\mathbb{R}^N_+)$, from (6.3.7) and (6.3.8), we get

$$\begin{aligned} \int_{\mathbb{R}^N_+} uD^\alpha\psi dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N_+} uD^\alpha(\eta_n\psi)dx = -\lim_{n\to\infty}\int_{\mathbb{R}^N_+}(\eta_n\psi)D^\alpha u dx \\ &= -\int_{\mathbb{R}^N_+}\psi D^\alpha u dx. \end{aligned}$$

(i.3) Finally, by combining (6.3.5) with the last equality and changing the variable again, we obtain

$$\begin{aligned} \langle D^\alpha U, \varphi\rangle &= \int_{\mathbb{R}^N_+}\psi D^\alpha u dx \\ &= \int_{\mathbb{R}^N_+}\varphi D^\alpha u dx + (-1)^{\alpha_N}\int_{\mathbb{R}^N_+}\varphi(x', -x_N)D^\alpha u(x', x_N)dx \\ &= \int_{\mathbb{R}^N_+}\varphi D^\alpha u dx + \int_{\mathbb{R}^N_-}\varphi(x', x_N)D^\alpha u(x', -x_N)dx \\ &= \langle E(D^\alpha u), \varphi\rangle. \end{aligned}$$

(ii) Hence $D^\alpha(Eu) = E(D^\alpha u)$, which implies $\|D^\alpha(Eu)\|_{L^p(\mathbb{R}^N)} \leq 2^{1/p}\|u\|_{W^{1,p}(\mathbb{R}^N_+)}$. Hence $Eu \in W^{1,p}(\mathbb{R}^N)$ and E is continuous from $W^{1,p}(\mathbb{R}^N_+)$ to $W^{1,p}(\mathbb{R}^N)$. □

The following theorem is analogous of Theorem 6.3.4 in $H^s(\Omega)$.

Theorem 6.3.9 *Assume $\Omega \subset \mathbb{R}^N$ is an open set satisfying H^s extension property, $s - \frac{N}{2} \in \mathbb{R}_+\backslash\mathbb{N}$, $m = \lfloor s - \frac{N}{2}\rfloor$ and $\sigma \in (0, s-m)$. Then $H^s(\Omega) \hookrightarrow C_b^{m,\sigma}(\overline{\Omega})$. In particular, there exists $C > 0$ such that for all $\alpha \in \mathbb{N}_0^N$, $|\alpha| = m$, and $x, y \in \Omega$, we have*

$$|D^\alpha u(x) - D^\alpha u(y)| \leq C\|D^\alpha u\|_{H^{s-m}(\Omega)}|x-y|^\sigma. \tag{6.3.9}$$

Furthermore, if Ω is bounded then $H^s(\Omega) \overset{c}{\hookrightarrow} C_b^m(\overline{\Omega})$.

Proof. Let $E : H^s(\Omega) \mapsto H^s(\mathbb{R}^N)$ be an extension operator and $R : H^k(\mathbb{R}^N) \mapsto H^k(\Omega)$ be the restriction operator (see Definition 6.3.6). From Theorem 6.3.4 and the continuity of operators E and R, we have the following continuous embeddings:

$$H^s(\Omega) \overset{E}{\hookrightarrow} H^s(\mathbb{R}^N) \hookrightarrow C_b^{m,\sigma}(\mathbb{R}^N) \overset{R}{\hookrightarrow} C_b^{m,\sigma}(\overline{\Omega}).$$

These embeddings in combination with estimate (6.3.3) prove the embedding $H^s(\Omega) \overset{c}{\hookrightarrow} C_b^m(\Omega)$ and the estimate (6.3.9). Furthermore, if Ω is bounded then $C_b^{m,\sigma}(\overline{\Omega}) \overset{c}{\hookrightarrow} C_b^m(\overline{\Omega})$ follows from Arzela-Ascoli lemma.

Theorem 6.3.10 *Let $k \in \mathbb{N}$ and assume Ω is an open bounded set that satisfies the H^{k+1} extension property. Then $H^{k+1}(\Omega) \overset{c}{\hookrightarrow} H^k(\Omega)$. In particular, $H_0^{k+1}(\Omega) \overset{c}{\hookrightarrow} H^k(\Omega)$ with Ω only open bounded.*

Proof. For the proof of the first part, see Theorem 9.5.2. The second part follows from Lemma 9.5.6 and the first part of this theorem. □

Remark 6.3.11 *Embedding theorems hold in a more general setting; see, for example, [1, 14, 44, 51]. Namely, we have the following embeddings, where $s > 0$, $p \in [1, \infty)$, $m \in \mathbb{N}$:*

		$\Omega = \mathbb{R}^N$, *or* $\Omega \subsetneq \mathbb{R}^N$ *satisfies* $W^{s,p}$ *extension property*	Ω *Lipschitz bounded*
a)	$s - \frac{N}{p} < 0$ $\frac{1}{\hat{p}} = \frac{1}{p} - \frac{s}{N}$	$W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, *for all* $q \in [p, \hat{p}]$	$W^{s,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$, *for all* $q \in [1, \hat{p})$
b)	$s - \frac{N}{p} = 0$	$W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, *for all* $q \in [p, \infty)$	$W^{s,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$, *for all* $q \in [1, \infty)$
c)	$s - \frac{N}{p} = m + \hat{\sigma}$ $\hat{\sigma} \in (0, 1)$	$W^{s,p}(\Omega) \hookrightarrow C_b^{m,\sigma}(\overline{\Omega})$, *for all* $\sigma \in [0, \hat{\sigma}]$	$W^{s,p}(\Omega) \overset{c}{\hookrightarrow} C_b^{m,\sigma}(\overline{\Omega})$, *for all* $\sigma \in [0, \hat{\sigma})$
d)	$s - \frac{N}{p} = m$	$W^{s,p}(\Omega) \hookrightarrow C_b^{m-1,\sigma}(\overline{\Omega})$, *for all* $\sigma \in [0, 1)$	$W^{s,p}(\Omega) \overset{c}{\hookrightarrow} C_b^{m-1,\sigma}(\overline{\Omega})$, *for all* $\sigma \in [0, 1)$
e)	$s - \frac{N}{p} \geq \sigma - \frac{N}{q}$ $p \leq q$	$W^{s,p}(\Omega) \hookrightarrow W^{\sigma,q}(\Omega)$	$W^{s+\epsilon,p}(\Omega) \overset{c}{\hookrightarrow} W^{\sigma,q}(\Omega)$, *for all* $\epsilon > 0$.

Furthermore

f) the continuous embeddings hold in $W_0^{s,p}(\Omega)$ instead of $W^{s,p}(\Omega)$, with Ω only open,

g) the compact embeddings hold with $W_0^{s,p}(\Omega)$ instead of $W^{s,p}(\Omega)$, with Ω only open bounded.

The following remark shows that Lipschitz regularity of Ω in Theorem 6.3.7 is essential.

Remark 6.3.12 *Theorem 6.3.7 does not hold in general if Ω is not Lipschitz. For example, let $\Omega = \{(x_1, x_2),\ x_1 \in (0,1),\ x_2 < x_1^\alpha\}$ with $\alpha > 1$. Clearly, Ω is not Lipschitz because it has a cusp at the origin.*

Let $u(x) = x_1^{-\frac{\sigma}{p}}$, $p \geq 1$, $\sigma > 0$. Given furthermore $m \in \mathbb{N}$, we can choose α such that $\alpha - \sigma - mp + 1 > 0$. Then $u \in W^{m,p}(\Omega)$ because for all integer $k \leq m$, we have

$$\begin{aligned} \int_\Omega |\partial_{x_1}^k u|^p dx &= C(k,p,\sigma) \int_0^1 \int_0^{x_1^\alpha} \left| x_1^{-\frac{\sigma}{p}-k} \right|^p dx_2 dx_1 \\ &= C(k,p,\sigma) \int_0^1 x_1^{\alpha-\sigma-kp} dx_1 = \frac{C(k,p,\sigma)}{\alpha - \sigma - kp + 1} < \infty. \end{aligned}$$

However, $u \notin L^\infty(\Omega)$. This means that Theorem 6.3.7 does not hold because if it does, from the embedding $H^2(\mathbb{R}^2) \hookrightarrow C_b^0(\mathbb{R}^2)$ (see Theorem 6.3.3 with $s = 2$, $p = 2$, $N = 2$ so $s - N/p = 1 > 0$), we should have $Eu \in C^0(\mathbb{R}^2)$.

Example 6.3.13 *Let $\Omega = (0,1) \subset \mathbb{R}$ and $u(x) = x$. Then $u \notin H_0^1(\Omega)$. Indeed, if we assume that $u \in H_0^1(\Omega)$, then there exists $u_n \in \mathcal{D}(\Omega)$ such that $u_n \to u$ in $H^1(\Omega)$. Note that we have $H_0^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$ because here $s - N/p = 1 - 1/2 = 1/2 > 0$. Therefore, $\|u - u_n\|_{C^0(\overline{\Omega})} \leq C\|u - u_n\|_{H^1(\Omega)}$ so $\|u - u_n\|_{C^0(\overline{\Omega})} \to 0$ as $n \to \infty$. But $u(1) = 1$ and $u_n(1) = 0$ for all n, so $\|u - u_n\|_{C^0(\overline{\Omega})} \geq |u(1) - u_n(1)| = 1$, which contradicts $\|u - u_n\|_{C^0(\overline{\Omega})} \to 0$ and proves the claim.*

Theorem 6.3.14 *Assume $\Omega \subset \mathbb{R}^N$ is an open set that satisfies the H^s extension property, $s > 0$. Then $C_b^\infty(\overline{\Omega}) \overset{d}{\hookrightarrow} H^s(\Omega)$. More generally, if $\Omega \subset \mathbb{R}^N$ is an open set that satisfies the $W^{s,p}$ extension property, $s > 0$, $p \in [1, \infty)$ then $C_b^\infty(\overline{\Omega}) \overset{d}{\hookrightarrow} W^{s,p}(\Omega)$.*

Proof. For the first part, let $u \in H^s(\Omega)$ and $U \in H^s$, $U = Eu$, where E is a $H^s(\Omega)$ extension operator. From Theorem 6.3.2, there exists a sequence (U_n) in $\mathcal{D}(\mathbb{R}^N)$ converging to U in $H^s(\mathbb{R}^N)$. It follows that (u_n), $u_n = RU_n$, is in $C_b^\infty(\overline{\Omega})$ and converges to u in $H^s(\Omega)$.

The proof for $W^{s,p}(\Omega)$ spaces is similar: extension in $W^{s,p}$, next using the embedding $\mathcal{D}(\mathbb{R}^N) \overset{d}{\hookrightarrow} W^{s,p}(\mathbb{R}^N)$ and conclude with the restriction to $W^{s,p}(\Omega)$; see [24].

6.4 Boundary traces in Sobolev spaces

Every $u \in W^{k,p}(\Omega)$ is defined almost everywhere, and in general is not a regular function in the classical sense, for example $u \notin C^0(\overline{\Omega})$. Therefore, a priori, it does not make sense to talk about the restriction of u on $\partial\Omega$. Trace theorems explain in what sense the boundary values of functions in Sobolev spaces are understood, and they provide important qualitative and quantitative properties of boundary traces. These theorems are important when studying PDEs equipped with boundary conditions.

Definition 6.4.1 *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. The restriction of $u \in \mathcal{S}(\overline{\Omega}) := \{U|_\Omega,\ U \in \mathcal{S}\}$ on $\partial\Omega$ is called "trace of u on $\partial\Omega$" and will be denoted by $\gamma(u)$. The map $u \in \mathcal{S}(\overline{\Omega}) \mapsto \gamma(u)$ is called a "trace operator".*

Trace theorem in Sobolev spaces essentially extends the operator γ in $H^s(\Omega)$ spaces (or more generally in $W^{s,p}(\Omega)$). Typically, the boundary trace of $H^s(\Omega)$ functions is constructed as follows. First, one defines the trace of $H^s(\mathbb{R}^N)$ functions on $\{x_N = 0\}$ by using Fourier transform. Next, for Ω general, the boundary trace of $H^s(\Omega)$ functions is constructed by using a partition of unity of $\partial\Omega$, see Definition 1.2.1, and the trace results on $\{x_N = 0\}$. Traces of $W^{s,p}(\Omega)$ functions are constructed similarly, but the proofs are substantially more difficult and technical; see, for example, [14, 33].

Theorem 6.4.2 *Let $s - \frac{1}{2} > 0$. The trace operator $\gamma : \mathcal{S}(\mathbb{R}^N) \mapsto \mathcal{S}(\mathbb{R}^{N-1})$, defined by $\gamma(u)(x') = u(x', 0)$, $x' \in \mathbb{R}^{N-1}$, extends to a linear continuous operator denoted with the same letter γ, from $H^s(\mathbb{R}^N)$ onto $H^{s-\frac{1}{2}}(\mathbb{R}^{N-1})$.*

Proof. For $x, \xi \in \mathbb{R}^N$, we will write $x = (x', x_N)$, $\xi = (\xi', \xi_N)$ with $x', \xi' \in \mathbb{R}^{N-1}$. Let $U \in \mathcal{S}(\mathbb{R}^N)$ and set $u(x') = U(x', 0)$, $u_N(x_N; x') = U(x', x_N)$ (u_N is considered as a function of x_N). Note that $u \in \mathcal{S}(\mathbb{R}^{N-1})$ and $u_N \in \mathcal{S}(\mathbb{R})$. Furthermore, if

$$\hat{U}(\xi) = \mathcal{F}_x[U](\xi), \quad \textit{resp.}\ \ \hat{u}(\xi') = \mathcal{F}_{x'}[u](\xi'), \quad \hat{u}_N(\xi_N; x') = \mathcal{F}_{x_N}[u_N(\cdot; x')](\xi_N)$$

is the Fourier transform of U in $\mathbb{R}^N$, resp. of u in $\mathbb{R}^{N-1}$, of u_N in $\mathbb{R}$, then

$$\hat{u}(\xi') = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \hat{U}(\xi', \xi_N) d\xi_N.$$

Indeed, this follows from Fourier and Fourier inversion formulas (5.4.3), (5.4.11) and the following equalities:

$$\begin{aligned}
\hat{u}_N(\xi_N; x') &= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-i(\xi_N x_N)} U(x', x_N) dx_N, \\
u(x') &= u_N(0; x') = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{i(0\xi_N)} \hat{u}_N(\xi_N; x') d\xi_N, \\
\hat{u}(\xi') &= \frac{1}{(2\pi)^{(N-1)/2}} \int_{\mathbb{R}^{N-1}} e^{-i(\xi' \cdot x')} u(x') dx' \\
&= \frac{1}{(2\pi)^{\frac{N-1}{2}}} \int_{\mathbb{R}^{N-1}} e^{-i(\xi' \cdot x')} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{i(0\xi_N)} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-i(\xi_N x_N)} U(x', x_N) dx_N d\xi_N dx' \\
&= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \hat{U}(\xi', \xi_N) d\xi_N.
\end{aligned}$$

Using inequalities (9.4.3), (9.4.4), we can evaluate it as follows:

$$\|u\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{N-1})} = \int_{\mathbb{R}^{N-1}} \langle \xi' \rangle^{2\left(s-\frac{1}{2}\right)} |\hat{u}(\xi')|^2 d\xi'$$

$$
\begin{aligned}
& \leq C \int_{\mathbb{R}^{N-1}} \langle\xi'\rangle^{2\left(s-\frac{1}{2}\right)} \left| \int_{\mathbb{R}} \hat{U}(\xi', \xi_N) d\xi_N \right|^2 d\xi' \\
(\textit{using Hölder}) \quad & \leq C \int_{\mathbb{R}^{N-1}} \langle\xi'\rangle^{2\left(s-\frac{1}{2}\right)} \int_{\mathbb{R}} \langle\xi\rangle^{2s} |\hat{U}(\xi', \xi_N)|^2 d\xi_N \\
& \qquad \int_{\mathbb{R}} \langle\xi\rangle^{-2s} d\xi_N d\xi'. \qquad (6.4.1)
\end{aligned}
$$

For $s - \frac{1}{2} > 0$, we have

$$
\begin{aligned}
\int_{\mathbb{R}} \langle\xi\rangle^{-2s} d\xi_N &= \int_{\mathbb{R}} (\langle\xi'\rangle^2 + |\xi_N|^2)^{-s} d\xi_N = \langle\xi'\rangle^{-2s} \int_{\mathbb{R}} \left(1 + \left(\frac{\xi_N}{\langle\xi'\rangle}\right)^2\right)^{-s} d\xi_N \\
&= \langle\xi'\rangle^{-2\left(s-\frac{1}{2}\right)} \int_{\mathbb{R}} (1+r^2)^{-s} dr \\
&= \langle\xi'\rangle^{-2\left(s-\frac{1}{2}\right)} I_s, \qquad (6.4.2)
\end{aligned}
$$

where $I_s := \int_{\mathbb{R}} (1+r^2)^{-s} dr < \infty$. Then (6.4.1) implies

$$
\begin{aligned}
\|u\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{N-1})} &\leq C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \langle\xi\rangle^{2s} |\hat{U}(\xi', \xi_N)|^2 d\xi_N d\xi' \leq C \|\hat{U}\|^2_{L^2_s(\mathbb{R}^N)} \\
&\leq C \|U\|^2_{H^s(\mathbb{R}^N)}. \qquad (6.4.3)
\end{aligned}
$$

By density of $\mathcal{S}(\mathbb{R}^N)$ in $H^s(\mathbb{R}^N)$, see Theorem 6.3.2, the inequality (6.4.3) is valid for all $U \in H^s(\mathbb{R}^N)$ and this proves the theorem. $\square$

Remark 6.4.3 *The condition $s - \frac{1}{2} > 0$ in Theorem 6.4.2 is used to show that $I_s < \infty$. One can ask whether this condition is due to the technique we used or it is critical. Actually, this condition is indeed critical. One can prove that for $0 < s \leq \frac{1}{2}$, the trace operator cannot be extended to a continuous operator from $H^s(\mathbb{R}^N)$ to $L^2(\mathbb{R}^{N-1})$. Furthermore, for such s, the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$ is $H^s_0(\Omega)$; see [35].*

Now we address the question of defining the trace of $H^s(\Omega)$ functions on $\partial\Omega$. We expect that the range of the trace operator will be in a certain space $H^s(\partial\Omega)$. The following definition addresses a more general boundary space $W^{s,p}(\partial\Omega)$; see for example [17].

Definition 6.4.4 *Let $1 \leq m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ of class $C^{m-1,1}$ be open and bounded. Let $\{(G_i, \theta_i),\ i = 1, \dots, M\}$ be as in Definition 1.2.4, with $\{G_i,\ i = 1, \dots, M\}$ a finite open covering of $\partial\Omega$, with $\theta_i \in C^{m-1,1}(\overline{Q}; \overline{G}_i)$ a diffeomorphism such that $\theta_i(Q^+) = \Omega \cap G_i$, $\theta_i(Q^0) = \partial\Omega \cap G_i$; see Fig. 6.4.1. Furthermore, let $\{\varphi_i,\ i = 0, 1, \dots, M\}$ be a partition of unity subordinate to $\{G_i,\ i = 1, \dots, M\}$; see Theorem 1.2.2. For $0 < s \leq m$, $p \in [1, \infty)$ we define*

$$
W^{s,p}(\partial\Omega) = \{u : \partial\Omega \mapsto \mathbb{R},\ \overline{(\varphi_i u) \circ \theta_i} \in W^{s,p}(\partial\mathbb{R}^N_+),\ i = 1, \dots, M\}, \qquad (6.4.4)
$$

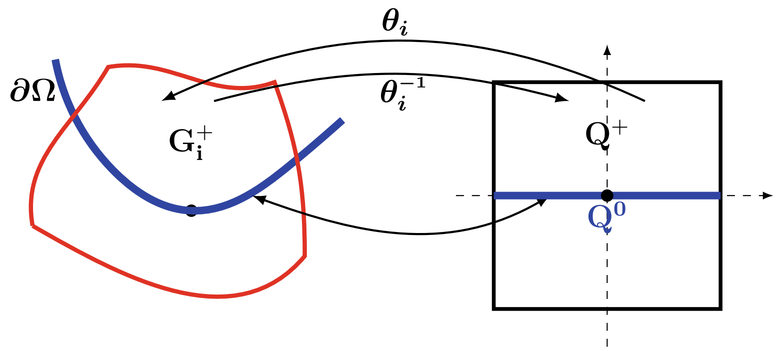

Figure 6.4.1: G_i^+, Q^+, θ_i, and θ_i^{-1}

and its norm by

$$\|u\|_{W^{s,p}(\partial\Omega)} = \sum_{i=1,M} \|\overline{(\varphi_i u)\circ\theta_i}\|_{W^{s,p}(\partial\mathbb{R}^N_+)},$$

where $\overline{(\varphi_i u)\circ\theta_i}$ *is the extension of* $(\varphi_i u)\circ\theta_i$ *in* $\partial\mathbb{R}^N_+$ *by zero.*

In the same spirit as in Remark 1.2.8, one can show that $\|\cdot\|_{W^{s,p}(\partial\Omega)}$ norms corresponding to two different partitions of unity of $\partial\Omega$ are equivalent; see also [17, 46]. We have this general result.

Theorem 6.4.5 *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set of class* $C^{0,1}$, $s > 0$, *and* $p \in (1,\infty)$.

i) Case $s - \dfrac{1}{p} \in (0,1)$. *There exists a linear continuous trace operator with continuous right inverse,*

$$\gamma : W^{s,p}(\Omega) \mapsto W^{s-\frac{1}{p},p}(\partial\Omega). \tag{6.4.5}$$

ii) Case $s - \dfrac{1}{p} \in [1,\infty)$. *Assume furthermore that* Ω *is of class* $C^{m,1}$, $m \geq 1$. *Then for* $s - \dfrac{1}{p} \in (\ell-1, m+1)\backslash\mathbb{N}$, *if* $p \neq 2$, *and* $s - \dfrac{1}{p} \in (\ell-1, m+1)$, *if* $p = 2$, *for a certain* $\ell \in \mathbb{N}$, *there exists a linear continuous trace operator with continuous right inverse,*

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\ell-1}) : W^{s,p}(\Omega) \mapsto \prod_{i=0}^{\ell-1} W^{s-i-\frac{1}{p},p}(\partial\Omega), \tag{6.4.6}$$

such that for $U \in \mathcal{S}(\overline{\Omega})$, *if* $(u_0, u_1, \ldots, u_{\ell-1}) = \gamma(U)$ *then* $u_i(x) = \partial^i_\nu U(x)$, *for all* $x \in \partial\Omega$, $i = 0, \ldots, \ell-1$. *Here,* $\partial^i_\nu U(x)$ *means the i-th order normal derivative of* u, *i.e.* $\partial^i_\nu U(x) = \displaystyle\sum_{|\alpha|=i} \frac{i!}{\alpha!} D^\alpha U \nu^\alpha$, *where* ν *is the unit outward normal vector to* $\partial\Omega$.

For the proof of this theorem in the case $s \in \mathbb{N}$, see [40], and in the case $s \notin \mathbb{N}$ see [25, 37].

We note that the condition for $s - \dfrac{1}{p}$ in ii) is due to these facts: a) Besov space $B^s_{p,p}(\Omega)$ is equal to $W^{s,p}(\Omega)$ only if $(s,p) \in ((0,\infty)\backslash\mathbb{N}) \times (1,\infty)$ or $(s,p) \in \mathbb{N} \times \{2\}$ (see [24, 44]) and b) Claim ii) holds for all $s - \dfrac{1}{p} \in (\ell - 1, m+1)$ with $\prod_{i=0}^{\ell-1} W^{s-i-\frac{1}{p},p}(\partial\Omega)$ replaced by $\prod_{i=0}^{\ell-1} B^{s-i-\frac{1}{p}}_{p,p}(\partial\Omega)$; see [25, 37]. For more details on the boundary traces, see [24, 25, 33, 37, 39, 40, 46, 48].

The boundary trace results allow us to integrate by parts with Sobolev space functions, as in the case of smooth functions.

Theorem 6.4.6 (Integration by parts) *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\alpha \in \mathbb{N}_0^N$, $|\alpha| = 1$.*

i) If $u \in W_0^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$, where $p \in [1,\infty)$, $1/p + 1/q = 1$, then

$$\int_\Omega u D^\alpha v dx = -\int_\Omega v D^\alpha u dx, \quad |\alpha| = 1. \tag{6.4.7}$$

ii) If Ω is furthermore a Lipschitz bounded set then for $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$, with $p \in (1,\infty)$, $1/p + 1/q = 1$, we have

$$\int_\Omega u D^\alpha v dx = \int_{\partial\Omega} \gamma(u)\gamma(v)\nu^\alpha d\sigma - \int_\Omega v D^\alpha u dx, \tag{6.4.8}$$

where $\nu = (\nu_1, \ldots, \nu_N)$ is the unit outward normal vector on $\partial\Omega$.

Proof. To prove (6.4.7), let (u_n) in $\mathcal{D}(\Omega)$ such that $\lim_{n\to\infty} u_n = u$ in $W^{1,p}(\Omega)$. Then

$$\begin{aligned} \int_\Omega u D^\alpha v dx &= \lim_{n\to\infty} \int_\Omega u_n D^\alpha v dx = \lim_{n\to\infty} \langle u_n, D^\alpha v\rangle \\ &= -\lim_{n\to\infty} \langle v, D^\alpha u_n\rangle = -\lim_{n\to\infty} \int_\Omega v D^\alpha u_n dx = -\int_\Omega v D^\alpha u_n dx. \end{aligned}$$

To prove (6.4.8) we consider (u_n) and (v_n) in $C^\infty(\overline{\Omega})$ converging respectively to u in $W^{1,p}(\Omega)$ and v in $W^{1,q}(\Omega)$; see Theorem 6.3.14. Then

$$\begin{aligned} \int_\Omega u D^\alpha v dx &= \lim_{n\to\infty} \int_\Omega D^\alpha u_n v_n dx = \lim_{n\to\infty} \int_{\partial\Omega} u_n v_n \nu^\alpha d\sigma - \lim_{n\to\infty} \int_\Omega u_n D^\alpha v_n dx \\ &= \int_{\partial\Omega} \gamma(u)\gamma(v)\nu^\alpha d\sigma - \int_\Omega u D^\alpha v dx, \end{aligned}$$

where for the limit in the boundary integral we used the continuity of the trace operator.

6.5 Poincaré inequality

Poincaré inequality, which exists in a variety of forms, essentially states that in a certain subspace of $W^{1,p}$, the L^p norm of u is bounded uniformly by the L^p norm of ∇u. This makes the L^p seminorm an equivalent norm, which is useful in a number of problems.

Theorem 6.5.1 (Poincaré inequality) *Let Ω be an open bounded set in at least one direction, for example, $\Omega \subset \{(x_1, \ldots, x_N),\ |x_i| \le d < \infty\}$, for a certain $d > 0$ and, $i \in \{1, \ldots, N\}$. Then for $p \in [1, \infty)$ there exists $C = C(p, \Omega) > 0$ such that the following inequality, called "Poincaré inequality", holds*

$$\forall u \in W_0^{1,p}(\Omega), \quad \|u\|_{L^p(\Omega)} \ \le \ C|u|_{W^{1,p}(\Omega)}, \qquad C = pd. \tag{6.5.1}$$

Proof. By the density of $\mathcal{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, it is enough to prove (6.5.1) only for $u \in \mathcal{D}(\Omega)$. Integration by parts gives

$$\begin{aligned}
\|u\|^p_{L^p(\Omega)} = \int_\Omega \partial_i x_i |u(x)|^p dx &= \lim_{\epsilon \to 0} \int_\Omega \partial_i x_i (u^2 + \epsilon^2)^{p/2} dx \\
&= -\lim_{\epsilon \to 0} \int_\Omega p x_i (u^2 + \epsilon^2)^{(p-2)/2} u \partial_i u dx \\
&= -\lim_{\epsilon \to 0} \int_\Omega p x_i (u^2 + \epsilon^2)^{(p-1)/2} \frac{u}{(u^2+\epsilon^2)^{1/2}} \partial_i u dx \\
\textit{(use Lebesgue DCT)} \quad &= -\int_\Omega p x_i |u|^{p-1} \mathrm{sgn}(u) \partial_i u dx \\
\textit{(use Hölder inequality)} \quad &\le pd \| |u|^{p-1} \|_{L^{\frac{p}{p-1}}(\Omega)} \|\partial_i u\|_{L^p(\Omega)}; \quad \textit{hence} \\
\|u\|_{L^p(\Omega)} &\le (pd) \|\partial_i u\|_{L^p(\Omega)},
\end{aligned} \tag{6.5.2}$$

where $\mathrm{sgn}(x) = 2H(x) - 1$, because $\| |u|^{p-1} \|_{L^{\frac{p}{p-1}}(\Omega)} = \|u\|^{p-1}_{L^p(\Omega)}$, which proves (6.5.1). □

We note that Poincaré inequality (6.5.1) can be generalized. For example, if Ω is bounded in one direction, then for $k \in \mathbb{N}$, $p \in [1, \infty)$ we have

$$\forall u \in W_0^{k,p}(\Omega), \quad \|u\|_{L^p(\Omega)} \le C|u|_{W^{k,p}(\Omega)}, \quad C = C(k, p, \Omega). \tag{6.5.3}$$

One can prove the inequality by applying (6.5.1) to all $D^\alpha u$, $\alpha \in \mathbb{N}_0^N$, $|\alpha| \le k - 1$.
Inequality (6.5.1) implies the following straightforward result.

Corollary 6.5.2 *Let $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^N$ be an open set bounded in at least one direction. Then $|u|_{W^{1,p}(\Omega)}$ is a norm in $W_0^{1,p}(\Omega)$, equivalent to $\|u\|_{W^{1,p}(\Omega)}$. Furthermore, when $p = 2$, $(u, v)_{H^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v dx$ defines an inner product equivalent to the usual inner product (6.1.5).*

Example 6.5.3 *Assume $\Omega \subset \mathbb{R}^N$ is a Lipschitz open bounded connected set and let*

$$M_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega), \ \int_\Omega u dx = 0 \right\}, \quad p \in [1, \infty).$$

Then Poincaré inequality (6.5.1) holds in $M_0^{1,p}(\Omega)$. Let us prove the claim for $p = 2$. Assume the claim does not hold. So

$$\forall n \in \mathbb{N}, \quad \exists u_n \in M_0^{1,2}(\Omega), \quad \|u_n\|_{L^2(\Omega)} \geq n \|\nabla u_n\|_{L^2(\Omega)}. \tag{6.5.4}$$

Dividing (6.5.4) by $\|u_n\|_{L^2(\Omega)}$ and setting $v_n = \dfrac{u_n}{\|u_n\|_{L^2(\Omega)}}$ we get for all n:

$$v_n \in M_0^{1,2}(\Omega), \quad 1 = \|v_n\|_{L^2(\Omega)} \geq n \|\nabla v_n\|_{L^2(\Omega)}. \tag{6.5.5}$$

So, the sequence (v_n) is bounded in $H^1(\Omega)$. As $H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$, there exists a subsequence of (v_n) still denoted by (v_n), converging in $L^2(\Omega)$ to a certain $v \in H^1(\Omega)$. From (6.5.5), we get $\nabla v_n \to 0$ in $L^2(\Omega)$, which implies that $v_n \to v$ in $H^1(\Omega)$ and $\nabla v = 0$. Necessarily we have

- *i)* *v is constant because $\nabla v = 0$,*
- *ii)* *$v \in M_0^{1,2}(\Omega)$ because from $v_n \in M_0^{1,2}(\Omega)$ we have $\int_\Omega v_n dx = 0$, which from the convergence $v_n \to v$ in $L^2(\Omega)$ implies $v \in M_0^1(\Omega)$,*
- *iii)* *$\|v\|_{L^2(\Omega)} = 1$ because passing to the limit in $\|v_n\|^2_{L^2(\Omega)} = 1$ implies $\|v\|_{L^2(\Omega)} = 1$.*

But then we have a contradiction because from i) and ii) we get $C = 0$, and then iii) becomes impossible. This proves that Poincaré inequality holds in $M_0^{1,2}(\Omega)$.

6.6 $H^{-s}(\Omega)$ and $W^{-s,q}(\Omega)$ spaces

When studying PDEs, often one has to consider elements of $(H^k(\Omega))'$, where in general E' is the topological dual space of the Banach space E. One would wonder whether elements of $(H^k(\Omega))'$ could be characterized by elements of $\mathcal{D}'(\Omega)$ which are continuous for the $H^k(\Omega)$ norm. The following lemma shows that $\mathcal{D}(\Omega)$ is not dense in $H^k(\Omega)$. It implies that if $\ell \in (H^k(\Omega))'$ then ℓ cannot be characterized by any $T \in \mathcal{D}'(\Omega)$ continuous for the $H^k(\Omega)$ norm.

Roughly speaking, negative order Sobolev spaces $H^{-s}(\Omega)$ are subspaces of $(H^s(\Omega))'$, elements of which are characterized by elements of $\mathcal{D}'(\Omega)$, continuous for the $H^s(\Omega)$ norm.

Lemma 6.6.1 *Let $\Omega \subset \mathbb{R}^N$ be open, bounded. The set $\mathcal{D}(\Omega)$ is not dense in $H^k(\Omega)$, $k \in \mathbb{N}$.*

Proof. Let $0 \neq u \in C^{2k}(Q)$ with $Q = (-a, a)^N$ such that $\Omega \Subset Q$. Clearly $u \in H^k(\Omega)$. Define

$$L \quad : \quad H^k(\Omega) \mapsto \mathcal{D}'(\Omega),$$

$$\langle Lu, \varphi\rangle := \left\langle \sum_{|\alpha|\le k} (-1)^{|\alpha|} D^{2\alpha} u, \varphi \right\rangle = \sum_{|\alpha|\le k} \int_\Omega D^\alpha u D^\alpha \varphi dx, \quad \varphi \in \mathcal{D}(\Omega).$$

If u is chosen such that

$$\sum_{|\alpha|\le k} (-1)^{|\alpha|} D^{2\alpha} u = 0 \quad in \;\; \Omega, \tag{6.6.1}$$

then $Lu = 0$. Note that (6.6.1) has nonzero solutions, for example $u(x) = \mathrm{Re}(e^{\lambda x_1})$, with $\lambda \neq 0$ solution to $\sum_{0\le\alpha_1\le k} (-1)^{\alpha_1} \lambda^{2\alpha_1} = 0$.

Now we can prove that $\mathcal{D}(\Omega)$ is not dense in $H^k(\Omega)$. If we assume $\mathcal{D}(\Omega)$ is dense in $H^k(\Omega)$ then there exists $\varphi_n \in \mathcal{D}(\Omega)$, $\lim\limits_{n\to\infty} \varphi_n = u$ in $H^k(\Omega)$. Therefore

$$0 = \lim_{n\to\infty} \langle Lu, \varphi_n\rangle = \lim_{n\to\infty} \sum_{|\alpha|\le k} \int_\Omega D^\alpha u D^\alpha \varphi_n dx = \sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^2 dx > 0,$$

which is a contradiction and proves the lemma. □

This lemma with the comments preceding it motivates the following definition.

Definition 6.6.2 *Let $s > 0$, $p \in [1, \infty)$, $q \in (1, \infty]$, $1/p + 1/q = 1$, and $\Omega \subset \mathbb{R}^N$. We define*[6] *$W^{-s,q}(\Omega)$ as the dual space of $W_0^{s,p}(\Omega)$. If Ω is $\partial\mathbb{R}^N_+$ or a $C^{\lfloor s\rfloor+1}$ domain, then $W^{-s,q}(\partial\Omega)$ denotes the dual space of $W^{s,p}(\partial\Omega)$. When $p = 2$, we write $H^{-s}(\cdot)$ instead of $W^{-s,2}(\cdot)$.*

Note that the spaces $W^{-s,q}(\Omega)$ and $W^{-s,q}(\partial\Omega)$ equipped with their dual norms are Banach spaces. Note also that $p = \infty$ is excluded from the previous definition, because the closure of $\mathcal{D}(\Omega)$ in $W^{s,\infty}(\Omega)$ is a subspace of $C^{\lfloor s\rfloor}(\overline{\Omega})$.

Furthermore, $\mathcal{D}(\Omega) \xhookrightarrow{d} W_0^{s,p}(\Omega)$ implies that $W^{-s,q}(\Omega)$ can be identified by a subspace of distributions in $\mathcal{D}'(\Omega)$ continuous with respect to $\|\cdot\|_{W^{s,p}(\Omega)}$ norm, which is not the case for the dual space of $W^{s,p}(\Omega)$.

Remark 6.6.3 *Let $s > 0$. In analogy to the characterization of $H^s(\mathbb{R}^N)$ by $L^2_s(\mathbb{R}^N)$, we have $H^{-s}(\mathbb{R}^N) = (H^s(\mathbb{R}^N))'$, i.e.*

$$H^{-s}(\mathbb{R}^N) = \left\{ \ell \in \mathcal{S}', \quad \|\ell\|^2_{H^{-s}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \langle\xi\rangle^{-2s} |\mathcal{F}[\ell]|^2 d\xi =: \|\hat{\ell}\|^2_{L^2_{-s}} < \infty \right\}.$$

For the proof see for example [12, 35].

[6] See Remark 6.6.3 for the motivation of the term $-s$.

Problems

Problem 6.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set, $u \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $p \in [1,\infty]$, and $\tilde{u}$ a function in Ω such that the N-dimensional Lebesgue measure of the set $\{x \in \mathbb{R}^N,\ \tilde{u}(x) \neq u(x)\}$ is zero. Prove that $\tilde{u} \in W^{k,p}(\Omega)$ and $\|\tilde{u}\|_{W^{k,p}(\Omega)} = \|u\|_{W^{k,p}(\Omega)}$.*

Problem 6.2 *Let $H = \mathbb{1}_{(0,\infty)}$ be the Heaviside function. Prove that $H \notin W^{1,p}(\Omega)$, where $p \in [1,\infty]$, $\Omega \subset \mathbb{R}$ is an open set, $0 \in \Omega$.*

Problem 6.3 *Let B be the unit ball in $\mathbb{R}^N$. Prove that $u(x) = \ln(1 - \ln|x|) \in W^{1,p}(B)$ iff $p < N$.*

Problem 6.4 *Let $h : \mathbb{R} \mapsto \mathbb{R}$ be a C^1 piecewise smooth function, i.e. h is C^0 in $\mathbb{R}$, $h(0) = 0$, $h' \in L^\infty(\mathbb{R})$, and h' is continuous in $\mathbb{R}$ except in a finite number of points Z where side derivatives exist. Prove that $h \circ u \in H^1(\Omega)$, $\Omega \subset \mathbb{R}^N$ open, for every $u \in H^1(\Omega)$, and*

$$\partial_i(h \circ u) = \mathbb{1}_{\{x\in\Omega,\ u(x)\notin Z\}} h'(u)\partial_i u, \quad i = 1, \dots, N.$$

Problem 6.5 *Assume $u \in H^1(\mathbb{R})$ and set $u_h(x) = \frac{1}{h}(u(x+h) - u(x))$. Prove that $\lim\limits_{h\to 0} u_h = u'$ in $L^2(\mathbb{R})$.*

Problem 6.6 *Find where, and explain why, Ω in Lemma 6.6.1 is required to be bounded. Prove that, in general, $\mathcal{D}(\Omega)$ is not dense in $H^1(\Omega)$ if $\Omega \subsetneq \mathbb{R}^N$ is unbounded.*

Problem 6.7 *Show that $H^1_0(\mathbb{R}\backslash\{0\}) \subsetneq H^1(\mathbb{R})$.*

Problem 6.8 *Let $N \geq 3$, $u \in H^1(\mathbb{R}^N)$ and $f(t,x) = (f_i(t,x))$, $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, where $f_i(t,x) = \partial_i u + tu\dfrac{x_i}{|x|^2}$. Show that*

$$0 \leq \int_{\mathbb{R}^N} |f(t,x)|^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx + (t^2 - (N-2)t) \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx,$$

and deduce that

$$\frac{u}{|x|} \in H^1(\mathbb{R}^N) \quad \text{and} \quad \left\| \frac{u}{|x|} \right\|_{L^2(\mathbb{R}^N)} \leq \frac{2}{N-2} \|\nabla u\|_{L^2(\mathbb{R}^N)}.$$

As a corollary of this result, prove[7] that $H^1_0(\mathbb{R}^N\backslash\{0\}) = H^1(\mathbb{R}^N)$.

Problem 6.9 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set, $s > 0$, $p \in [1,\infty)$, and define*

$$\tilde{W}^{s,p}(\Omega) = \Big\{ u : \quad \|u\|_{\tilde{W}^{s,p}(\Omega)} := \inf\{\|U\|_{W^{s,p}(\mathbb{R}^N)},\ \ U|_\Omega = u\} \Big\}.$$

Show that $W^{s,p}(\Omega) = \tilde{W}^{s,p}(\Omega)$ and the norms $\|\cdot\|_{W^{s,p}(\Omega)}$, $\|\cdot\|_{\tilde{W}^{s,p}(\Omega)}$ are equivalent. Show furthermore that the statement does not hold if Ω is not bounded Lipschitz.

[7]This statement is true even for $N = 2$, but the proof is more delicate; see [51].

Problem 6.10 *Prove that there exist open bounds sets $\Omega \subset \mathbb{R}^N$ such that $C^1(\overline{\Omega})$ is not dense in $H^1(\Omega)$.*

Problem 6.11 *Prove Theorem 6.1.9.*

Problem 6.12 *Show that the embedding $H^{k+1}(\Omega) \hookrightarrow H^k(\Omega)$ is not compact, in general, if $\Omega \subset \mathbb{R}^N$ is only an open bounded set.*

Problem 6.13 *Let $u \in W^{1,p}(\Omega)$, $p \in [1, \infty]$, $\Omega = (a, b) \subset \mathbb{R}$. Prove that there exists $\tilde{u} \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, $\tilde{u} = u$ a.e. in Ω and*

$$\forall x, y \in \overline{\Omega}, \quad \tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(s)ds.$$

Problem 6.14 *Prove that if $u \in H^1(\mathbb{R})$ then u cannot have jump discontinuities.*

Problem 6.15 *Prove that $H^1(\mathbb{R}^N) \nsubseteq C^0(\mathbb{R}^N)$ for $N \geq 2$, i.e. there exists $u \in H^1(\mathbb{R}^N)$ such that $u \notin C^0(\mathbb{R}^N)$.*

Problem 6.16 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set and $s > r \geq 0$. Prove that $H^s(\Omega) \overset{c}{\hookrightarrow} H^r(\Omega)$.*

Problem 6.17 *Let $k \in \mathbb{N}$. Prove that the embedding $H^{k+1}(\mathbb{R}^N) \hookrightarrow H^k(\mathbb{R}^N)$ is not compact.*

Problem 6.18 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Prove that the embedding $H^1_0(\Omega) \hookrightarrow H^{-1}(\Omega)$ is compact.*

Problem 6.19 *Prove that Poincaré inequality does not hold in $H^1(\mathbb{R}^N)$.*

Problem 6.20 *Let $\Omega \subset \mathbb{R}^2$ be an open bounded Lipschitz connected set, $\gamma \subset \partial\Omega$ with nonzero length, and*

$$W^{1,p}_\gamma(\Omega) = \{u \in W^{1,p}(\Omega),\ u = 0 \text{ on } \gamma\}, \quad p \in [1, \infty).$$

Prove that Poincaré inequality (6.5.3) holds in $W^{1,p}_\gamma(\Omega)$.

Problem 6.21 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz connected set, $\omega \subset \Omega$ with nonzero measure, and*

$$W^{1,p}_\omega(\Omega) = \{u \in W^{1,p}(\Omega),\ u = 0 \text{ on } \omega\}, \quad p \in [1, \infty).$$

Prove that Poincaré inequality (6.5.3) holds in $W^{1,p}_\omega(\Omega)$.

Problem 6.22 *Let $k - \dfrac{N}{2} > 0$. Prove that $\delta_0 \in H^{-k}(\mathbb{R}^N)$.*

Problem 6.23 *Let $\Omega \subset \mathbb{R}^N$ be of class C^1, $g \in H^1(\Omega)$, and $\ell \in \mathcal{L}(H^1(\Omega))$ defined by $\ell(v) = \displaystyle\int_{\partial\Omega} g(x)v(x)d\sigma$, $v \in H^1(\Omega)$. Estimate the norm of ℓ seen as an element of $H^{-1}(\Omega)$ or of $\mathcal{L}(H^1(\Omega))$.*

7. Second-order linear elliptic PDEs: weak solutions

There exist several well-developed theories of elliptic linear PDEs of order two. We have already seen Perron's method for Laplace equation in Chapter 4, which provides classical solutions. In this context, Schauder theory provides $C^{2,\alpha}$ classical solutions; see, for example, [20].

Other methods, such as Hilbert spaces, variational, singular integral, or spectral methods provide the so-called "*weak solutions*", which belong to certain Sobolev spaces; see, for example, [5, 16, 20, 35].

In this chapter, we will only deal with the existence and uniqueness of weak solutions obtained by the variational method. By combining these results with some compactness results in Sobolev spaces, we will also address the solution to certain nonlinear elliptic PDEs.

7.1 Introduction

The objective of this chapter is to study the existence and uniqueness of a weak solution $u : \Omega \mapsto \mathbb{R}$ to the following problem:

$$Lu := -\sum_{i,j=1}^{N} \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^{N} b_i\partial_i u + cu = f \quad in \ \ \Omega, \tag{7.1.1a}$$

$$u = 0 \quad on \ \ \partial\Omega. \tag{7.1.1b}$$

This problem is commonly referred to as the "*Dirichlet problem*", which is in connection with the boundary condition $u = 0$. One may consider the general Dirichlet boundary condition $u = g$; see Remark 7.2.13.

Here and throughout this chapter, unless otherwise stated, $\Omega \subset \mathbb{R}^N$ is an open bounded set, and a_{ij}, b_i, and c satisfy

$$A := (a_{ij}) = (a_{ji}) \in W^{1,\infty}(\Omega; \mathbb{R}^{N^2}), \quad b := (b_i) \in L^\infty(\Omega; \mathbb{R}^N), \quad c \in L^\infty(\Omega), \tag{7.1.2a}$$

$$\exists k, K > 0, \ \ \forall 0 \neq \xi \in \mathbb{R}^N, \ \ k|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(\cdot)\xi_i\xi_j \leq K|\xi|^2, \quad a.e. \ in \ \Omega. \tag{7.1.2b}$$

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6_7

The assumptions for f will be provided later in the context.

We note that even in the case when $L = -\Delta$, there are cases when the problem $Lu = 0$ in Ω, $u = g$ on $\partial\Omega$ does not have a classical solution; see problem (4.4.3). We will revisit this problem in Example 7.2.14 and show that it has a unique weak solution.

There are three main questions relative to (7.1.1).

$$\begin{cases} Q1. & \textit{How does one understand the solution to } (7.1.1)\,? \\ Q2. & \textit{Does } (7.1.1) \textit{ have a solution? If yes, is the solution unique?} \\ Q3. & \textit{How regular is the solution to } (7.1.1)\,? \end{cases} \tag{7.1.3}$$

In this chapter we will address the first two questions. The third question is a classical topic involving fine techniques and can be found in numerous PDE books, such as [5, 20]. For the ease of the reader, we have included the solution to Q3 in Section 9.6 in Annex.

Definition 7.1.1

i) Assume $f \in C^0(\overline{\Omega})$. A strong, or classical, solution to (7.1.1) is every $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfying (7.1.1) pointwise.

ii) Assume $f \in H^{-1}(\Omega)$. A weak solution to (7.1.1) is every $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \sum_{i,j=1}^N a_{ij}\partial_i u \partial_j v + \int_\Omega \sum_{i=1}^N b_i \partial_i u v + \int_\Omega cuv = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega). \tag{7.1.4}$$

The equation (7.1.4) is called "the weak form equation of (7.1.1)".

The motivation for (7.1.4) is as follows. The operator L defines a linear continuous map $L : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$. Indeed, for $u \in H_0^1(\Omega)$ define $Lu \in \mathscr{D}'(\Omega)$ by

$$\begin{aligned} \langle Lu, v\rangle &:= \sum_{i,j=1,N} \langle \partial_j(-a_{ij}\partial_i u), v\rangle + \sum_{i=1,N} \langle b_i \partial_i u, v\rangle + \langle cu, v\rangle \\ &= \int_\Omega \left(\sum_{i,j=1,N} a_{ij}\partial_i u \partial_j v + \sum_{i=1,N} b_i \partial_i u v + cuv \right) dx \\ &=: B(u,v). \end{aligned} \tag{7.1.5}$$

By density of $\mathscr{D}(\Omega)$ in $H_0^1(\Omega)$, (7.1.5) is well-defined for all $u, v \in H_0^1(\Omega)$ and satisfies

$$\langle Lu, v\rangle = B(u,v), \qquad |\langle Lu, v\rangle| \le C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \tag{7.1.6}$$

where $C = C(A, b, c)$, which proves $L \in \mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))$ and $B \in \mathcal{L}(H_0^1(\Omega) \times H_0^1(\Omega))$. Actually, B defines a continuous bilinear form in $H^1(\Omega)$, i.e. (7.1.6) holds for all $u, v \in H^1(\Omega)$. Therefore, (7.1.4) is nothing but $\langle Lu, v\rangle = B(u,v) = \langle f, v\rangle$. It solves (7.1.1a) in a weak sense, and satisfies (7.1.1b) in the sense of traces $\gamma(u) = 0$ because $u \in H_0^1(\Omega)$.

Finally, if u is a classical solution with $u \in C^1(\overline{\Omega})$, (7.1.5) is obtained by multiplying the left-hand side of (7.1.1a) by v, integrating by part, and canceling boundary integrals by using $u = 0$ on the boundary. Hence, (7.1.4) gives a necessary condition for a strong solution u.

7.2 Existence and uniqueness of weak solutions

The existence of solutions to second-order linear elliptic PDEs relies on certain results of Functional Analysis, such as Lax-Milgram lemma and the theory of (linear) compact operators, as well as on a weak maximum principle in the context of Sobolev spaces, which we present in the following section.

7.2.1 Preliminary results

We will give some results of Functional Analysis. For their proof, the interested reader can see, for example, [20, 29]. We will also give the weak maximum principle in Sobolev spaces context and some other auxiliary results.

Theorem 7.2.1 (Riesz representation theorem) *Let H be an Hilbert space with inner product $(\cdot,\cdot)_H$ and $\ell \in H'$. Then there exists a unique $u \in H$ such that*

$$\forall v \in H, \ \ \langle \ell, v\rangle_{H'\times H} = (u,v)_H, \quad \|u\|_H = \|\ell\|_{H'}. \tag{7.2.1}$$

The following two theorems are the main ingredients for solving (7.1.1). While Theorem 7.2.2 solves (7.1.1) in special cases, Theorem 7.2.3 gives a full characterization of the weak solutions to (7.1.1).

Theorem 7.2.2 (Lax-Milgram lemma) *Let H be a Hilbert space and $B : H \times H \mapsto \mathbb{R}$ be a bilinear form satisfying*

$$\text{(continuity)} \qquad |B(u,v)| \le K\|u\|_H\|v\|_H, \quad \forall u,v \in H, \tag{7.2.2a}$$
$$\text{(ellipticity)} \qquad k\|u\|_H^2 \le B(u,u), \qquad \forall u \in H, \tag{7.2.2b}$$

with certain $K, k > 0$. Then, for every $f \in H'$ there exists a unique $u \in H$ such that

$$B(u,v) = \langle f, v\rangle_{H'\times H}, \quad \forall v \in H, \tag{7.2.3}$$
$$\|u\|_H \le \frac{1}{k}\|f\|_{H'}. \tag{7.2.4}$$

Theorem 7.2.3 (Fredholm alternative) *Let H be an Hilbert space and $A : H \mapsto H$, a compact linear operator. Then all the following hold.*

(i) $N(I-A) = \{v \in H, \ (I-A)v = 0\}$, *the kernel of $I-A$, has finite dimension.*

(ii) $R(I-A) = (N(I-A^*))^\perp$, *where $R(I-A) = \{(I-A)v, \ v \in H\}$ is the "range of $I-A$", A^* is the "adjoint of A", defined by $(A^*u,v)_H = (u,Av)_H$, and $(N(I-A^*))^\perp = \{w \in H, \ (w,v)_H = 0, \ \forall v \in N(I-A^*)\}$.*

(iii) $R(I-A) = H$ *iff* $N(I-A) = \{0\}$.

(iv) $dim(N(I-A)) = dim(N(I-A^*))$.

The following definition gives a generalization of concepts of sub/super harmonic functions related to the operator L and defines the concept of inequality on the boundary in the context of Sobolev spaces.

Definition 7.2.4 *Let $u, w \in H^1(\Omega)$, $v \in H_0^1(\Omega)$, and $\langle Lu, v\rangle$ defined by (7.1.5).*[1]

i) We say $Lu \leq 0$ if $\langle Lu, v\rangle \leq 0$ for all $0 \leq v \in H_0^1(\Omega)$. Similarly, we say $Lu \geq 0$ if $\langle Lu, v\rangle \geq 0$ for all $0 \leq v \in H_0^1(\Omega)$.

ii) We say $u \leq w$ on $\partial\Omega$ if $(u-w)^+ := \max\{u-w, 0\} \in H_0^1(\Omega)$. Similarly, $u \geq w$ on $\partial\Omega$ if $(u-w)^- := \min\{u-w, 0\} \in H_0^1(\Omega)$.

Now we give the "*weak maximum principle*" associated with the operator L in the context of Sobolev spaces. It generalizes the classical weak maximum principle that we have seen in Chapter 4.

Theorem 7.2.5 (Weak maximum principle) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $c \geq 0$ and $u \in H^1(\Omega)$ such that $Lu \leq 0$ ($Lu \geq 0$). Then*

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u^+(x), \quad \left(\inf_{x\in\Omega} u(x) \geq \inf_{x\in\partial\Omega} u^-(x)\right),$$

where sup *and* inf *are taken for x almost everywhere in Ω or on $\partial\Omega$.*

Proof. Consider first the case $Lu \leq 0$. If $M = \sup_{x\in\partial\Omega} u^+(x)$ and $m = \sup_{x\in\Omega} u(x)$ we have to prove $m \leq M$. Assume the claim does not hold, i.e. $M < m$. The proof continues strongly using $\langle Lu, v\rangle \leq 0$ with some particular functions v, and Sobolev embeddings in Remark 6.3.11.

Let $r \in [M, m)$. From Example 6.1.14 we have $w := (u-r)^+ \in H^1(\Omega)$. Actually, $w \in H_0^1(\Omega)$ for a certain $r \in [M, m)$. Indeed, there exists $r \in [M, m)$ such that

$$\inf\{|x-y|,\ x \in \operatorname{supp}((u-r)^+),\ y \in \partial\Omega\} > 0,$$

because if not we would get $M = m$, which by assumption is excluded. If we denote still by w the extension of w in $\mathbb{R}^N$ by zero, and consider $w_n = w * \rho_n$, with (ρ_n) a mollifier sequence, then $w_n|_\Omega \in \mathcal{D}(\Omega)$, for n large, and converges to w in $H^1(\Omega)$. So $(u-r)^+ \in H_0^1(\Omega)$. Furthermore, $(u-s)^+ \in H_0^1(\Omega)$ for all $s \in [r, m)$, because $\operatorname{supp}((u-s)^+) \subset \operatorname{supp}((u-r)^+)$.

Now we take $v = (u-s)^+$, $s \in [r, m)$. As $\in H_0^1(\Omega)$ we have

$$\sum_{i,j=1}^N \int_\Omega a_{ij}\partial_i u \partial_j v = \langle Lu, v\rangle - \sum_{i=1}^N \int_\Omega b_i \partial_i u v - \int_\Omega cuv \leq C \int_\Omega |v||\nabla u|, \tag{7.2.5}$$

[1] Which is well-defined for all $u, v \in H^1(\Omega)$.

because $\langle Lu, v\rangle \le 0$. From (6.1.17) we have $\nabla v = (\nabla u)\mathbb{1}_{\{u>s\}}$. Then (7.2.5) implies

$$\sum_{i,j=1}^{N}\int_\Omega a_{ij}\partial_i v\partial_j v \le C\int_{\omega_s} v|\nabla v|, \quad \omega_s = \operatorname{supp}(\nabla v)\subset \operatorname{supp}(v),$$

which combined with (7.2.2b) yields $k\|\nabla v\|^2_{L^2(\Omega)} \le C\|v\|_{L^2(\omega)}\|\nabla v\|_{L^2(\Omega)}$, or equivalently

$$\|\nabla v\|_{L^2(\Omega)} \le C\|v\|_{L^2(\omega_s)}, \quad C = C(L,k,\Omega). \tag{7.2.6}$$

Now we use Sobolev embedding results, Remark 6.3.11, a) for $N\ge 3$ and b) for $N=2$. For $N\ge 3$ we obtain

$$\begin{aligned}
\|v\|_{L^{\frac{2N}{N-2}}(\omega_s)} &\le \|v\|_{L^{\frac{2N}{N-2}}(\Omega)} && \textit{(use Poincaré inequality,}\\
&\le C_1\|\nabla v\|_{L^2(\Omega)} && (7.2.6)\ \textit{and}\\
&\le C_2\left(\int_{\omega_s} 1v^2\right)^{1/2} && \textit{Hölder inequality (9.4.1))}\\
&\le C_2\left(\|1\|_{L^{N/2}(\omega_s)}\|v^2\|_{L^{\frac{N}{N-2}}(\omega_s)}\right)^{1/2}\\
&\le C_2|\omega_s|^{1/N}\|v\|_{L^{\frac{2N}{N-2}}(\omega_s)},
\end{aligned}$$

which gives $C^{-N}\le|\omega_s|$. For $N=2$ we obtain $C^{-p}\le|\omega|$, for any $p>2$. In both cases C does not depend on s. Setting $C_0=\min\{C^{-N},C^{-p}\}$, we get

$$0 < C_0 \le |\omega_s| = |\operatorname{supp}(\nabla(u-s)^+)| \subset \operatorname{supp}((u-s)^+), \quad s\in[r,m). \tag{7.2.7}$$

Letting s to m in (7.2.7) implies $m<\infty$, because, if not, from $u\ge s$ in ω_s we get $u\notin L^2(\Omega)$. Therefore (7.2.7) holds for $s=m$, which implies that u attains its finite supremum m in ω_m, where at the same time necessarily $\nabla u = 0$ and so $|\omega_m|=0$, because $u=m$ in ω_m (see (6.1.17), Example 6.1.14). This contradicts (7.2.7) for $s=m$ and proves the case $Lu\le 0$. The case $Lu\ge 0$ is treated by considering $L(-u)\le 0$. □

This theorem implies the following uniqueness result.

Corollary 7.2.6 *Let $c\ge 0$ and $u\in H^1_0(\Omega)$ be a weak solution to (7.1.1) with $f=0$. Then $u=0$. So, (7.1.4) cannot have more than one solution.*

We will see that in the case $c\ge 0$, the existence of a weak solution to (7.1.1) is proved easily by using Lax-Milgram lemma. The case c arbitrary is addressed by using Fredholm alternative theorem and the following lemma.

Lemma 7.2.7 *There exists $k_0,\ K_0>0$, and $\lambda_0\ge 0$ such that for all $u,v\in H^1(\Omega)$ we have*

$$|B(u,v)| \le K_0\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \tag{7.2.8}$$

$$k_0|u|^2_{H^1(\Omega)} \leq B(u,u) + \lambda_0\|u\|^2_{L^2(\Omega)}. \tag{7.2.9}$$

If $b = 0$ and $c \geq 0$ we can take $k_0 = k$ and $\lambda_0 = 0$. Otherwise $k_0 = \left(k - \epsilon\|b\|_{L^\infty(\Omega)}\right)$ and $\lambda_0 = (1/(4\epsilon))\|b\|_{L^\infty(\Omega)} + \|c^-\|_{L^\infty(\Omega)}$, where $\|b\|_{L^\infty(\Omega)} = \sum_{i=1}^N \|b_i\|_{L^\infty(\Omega)}$ and $\epsilon > 0$ small so that $k_0 > 0$.

Proof. Clearly, (7.2.8) follows by using Hölder inequality to all the terms of $B(u,v)$. For (7.2.9), clearly it holds with $\lambda_0 = 0$ if $b = 0$ and $c \geq 0$. Otherwise, from ellipticity condition (7.1.2b), by using (9.4.2) for $\epsilon > 0$ we have

$$\begin{aligned}
k|u|^2_{H^1_0(\Omega)} &\leq \sum_{i,j=1}^N \int_\Omega a_{ij}\partial_i u \partial_j u = B(u,u) - \sum_{i=1}^N \int_\Omega b_i u \partial_i u - \int_\Omega cu^2 \\
&\leq B(u,u) \\
&+ \|b\|_{L^\infty(\Omega)} \left(\frac{1}{4\epsilon}\|u\|^2_{L^2(\Omega)} + \epsilon\|\nabla u\|^2_{L^2(\Omega)}\right) \\
&+ \|c^-\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)}.
\end{aligned}$$

It implies

$$\begin{aligned}
\left(k - \epsilon\|b\|_{L^\infty(\Omega)}\right)|u|^2_{H^1_0(\Omega)} &\leq B(u,u) \\
&+ \left(\frac{1}{4\epsilon}\|b\|_{L^\infty(\Omega)} + \|c^-\|_{L^\infty(\Omega)}\right)\|u\|^2_{L^2(\Omega)},
\end{aligned}$$

which proves the lemma.

7.2.2 Dirichlet problem

We start with a simple existence result.

Theorem 7.2.8 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $f \in H^{-1}(\Omega)$, $b = 0$, and $c \geq 0$. Then (7.1.1) has a unique weak solution $u \in H^1_0(\Omega)$ which satisfies*

$$\|u\|_{H^1(\Omega)} \leq \frac{1}{k}\|f\|_{H^{-1}(\Omega)}. \tag{7.2.10}$$

Proof. Lemma 7.2.7 holds with $\lambda_0 = 0$. Then it is clear that $B(u,v)$ satisfies the conditions of Lax-Milgram lemma, Theorem 7.2.2, which proves the theorem. □

Example 7.2.9 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $c \in L^\infty(\Omega)$, and consider the problem*

$$-\Delta u + cu = f \quad in \ \ \Omega, \qquad u = 0 \quad on \ \ \partial\Omega. \tag{7.2.11}$$

The weak form equation of this problem is given by

$$B(u,v) = \int_\Omega (\nabla u \cdot \nabla v + cuv)dx, \quad u,v \in H^1_0(\Omega). \tag{7.2.12}$$

Clearly, if $c \geq 0$, Theorem 7.2.8 implies that (7.2.11) has a unique weak solution. In fact, one can prove that there exists $\epsilon_0 > 0$ such that if $\|c^-\|_{L^\infty(\Omega)} < \epsilon_0$ then still (7.2.11) has a unique weak solution. Indeed, using Poincaré inequality in $H^1_0(\Omega)$ we get

$$
\begin{aligned}
B(u,u) &= \int_\Omega (|\nabla u|^2 + (c^+ + c^-)u^2)dx \\
&\geq \int_\Omega |\nabla u|^2 dx \int_\Omega + c^- u^2 dx \\
&\geq \int_\Omega |\nabla u|^2 - \|c^-\|_{L^\infty(\Omega)} \int_\Omega u^2 dx \qquad (use\ (6.5.1)) \\
&\geq \int_\Omega |\nabla u|^2 - C^2 \|c^-\|_{L^\infty(\Omega)} |\nabla u|^2 \\
&= (1 - C^2\|c^-\|_{L^\infty(\Omega)}) |\nabla u|^2_{H^1_0(\Omega)}, \qquad (7.2.13)
\end{aligned}
$$

where C is the (Poincaré) constant in (6.5.1). Inequality (7.2.13) shows that B is elliptic in $H^1_0(\Omega)$ if $\|c^-\|_{L^\infty(\Omega)} < C^{-2}$. Then the existence and uniqueness of a weak solution follows from Lax-Milgram lemma.

In the case when $c \not\geq 0$ or $b \neq 0$, the issue of existence of solutions to (7.1.4) is more complex. We will solve it as follows. First we will consider an auxiliary problem $L_0 u = f$, $f \in H^{-1}(\Omega)$, $L_0 u = Lu + \lambda_0 u$, which, thanks to Lax-Milgram lemma, has a unique weak solution. The operator L_0 generates a map $G_0 : H^1_0(\Omega) \mapsto H^1_0(\Omega)$, which is compact. Then, the equation (7.1.4) is written equivalently in the form $(I - G_0)u = L_0^{-1} f$, which is analyzed by using Fredholm alternative Theorem 7.2.3.

More precisely, let $L_0 : H^1_0(\Omega) \mapsto H^{-1}(\Omega)$ and $B_0 \in \mathscr{L}(H^1_0(\Omega) \times H^1_0(\Omega))$ be given by

$$
L_0 u := Lu + \lambda_0 u = -\sum_{i,j=1}^N \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^N b_i \partial_i u + (c + \lambda_0)u, \qquad (7.2.14)
$$

$$
\begin{aligned}
B_0(u,v) &= \langle L_0 u, v\rangle_{H^{-1}\times H^1_0} \qquad (7.2.15)\\
&= B(u,v) + \lambda_0 (u,v)_{L^2(\Omega)} \\
&= \sum_{i,j=1}^N \int_\Omega a_{ij}\partial_i u \partial_j v + \sum_{i=1}^N \int_\Omega b_i \partial_i u v + \int_\Omega (c+\lambda_0) uv, \quad \forall v \in H^1_0(\Omega),
\end{aligned}
$$

where λ_0 is introduced in Lemma 7.2.7.

Before we address the solution to (7.1.1) in general, we present the following technical lemmas.

Lemma 7.2.10 *The equation $L_0 u = f$, $f \in H^{-1}(\Omega)$, has a unique weak solution $u \in H^1_0(\Omega)$ satisfying*

$$
\begin{aligned}
\langle L_0 u, v\rangle_{H^{-1}(\Omega)\times H^1(\Omega)} &= B_0(u,v) = \langle f, v\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)}, \quad \forall v \in H^1_0(\Omega), \\
\|u\|_{H^1(\Omega)} &\leq C\|f\|_{H^{-1}(\Omega)}. \qquad (7.2.16)
\end{aligned}
$$

Hence, the maps $L_0 : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ and $L_0^{-1} : H^{-1}(\Omega) \mapsto H_0^1(\Omega)$ are linear, invertible, and bounded.

Proof. Lemma 7.2.7 implies that B_0 is continuous and elliptic in $H_0^1(\Omega)$, and therefore the results follow from Lax-Milgram lemma. $\square$

The following lemma uses some properties of compact operators, and the reader can consult [29] for more details about compact operators.

Lemma 7.2.11 *Assume λ_0 of Lemma 7.2.7 is positive, and let $I : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ be the embedding map, $Iu = u$, and $G_0 := \lambda_0 L_0^{-1} \circ I : H_0^1(\Omega) \mapsto H_0^1(\Omega)$. Then G_0 is compact.*

Proof. Clearly G_0 is linear. For the compactness, we write $I = I_0 \circ I_1$, where $I_1 : H_0^1(\Omega) \mapsto L^2(\Omega)$, $I_0 : L^2(\Omega) \mapsto H^{-1}(\Omega)$, $I_1 u = I_0 u = u$. So $G_0 = \lambda_0 L_0^{-1} \circ I_0 \circ I_1$, with L_0^{-1}, I_0, I_1 continuous, and I_1 compact; see Theorem 6.3.10. Hence, G_0 is compact as a composition of a compact operator with a continuous operator. This proves (i).

Theorem 7.2.12 *If $c \geq 0$ then the problem (7.1.4) has a unique solution $u \in H_0^1(\Omega)$. Otherwise we have if $N(L) = \{0\}$ then (7.1.4) has a unique solution, and if $N(L) \neq \{0\}$ then (7.1.4) has a solution (and then a finite number of solutions) iff $f \in R(L)$.*

Proof. We note that (7.1.4), which we refer to as "$Lu = f$", is equivalent to $(I - G_0)u = f_0$ with $f_0 = L_0^{-1} f$. In the following part of the proof, we will use Theorem 7.2.3 with $H = H_0^1(\Omega)$ and $A = G_0$.

If $c \geq 0$ and $b = 0$, Theorem 7.2.8 shows that $Lu = f$ has a unique solution. If $c \geq 0$ and $b \neq 0$, necessarily $\lambda_0 > 0$. It implies $N(I - G_0) = \{0\}$, because $(I - G_0)u = 0$ is equivalent to $Lu = 0$, which from Corollary 7.2.6 implies $u = 0$. Then from (iii), Theorem 7.2.3, we get that $(I - G_0)u = f_0$ has a unique solution, so $Lu = f$ has a unique solution.

Otherwise, in the case $c \ngeq 0$ necessarily $\lambda_0 > 0$. If $N(I - G_0) = \{0\}$, like above, $Lu = f$ has a unique solution from (iii), Theorem 7.2.3. Finally, if $N(I - G_0) \neq \{0\}$, (ii) Theorem 7.2.3 implies that $(I - G_0)u = f_0$ has a solution (and in this case, a finite number of weak solutions) if and only if $f_0 \in N(I - G_0^*)^\perp$, which is equivalent to $f \in R(L)$ and completes the proof.

Remark 7.2.13 *Consider the problem*

$$\textit{find } u \in H^1(\Omega), \quad Lu = f \ \textit{ in } \ \Omega, \quad \gamma(u) = \gamma(g) \ \textit{ on } \ \partial\Omega, \tag{7.2.17}$$

where $f \in H^{-1}(\Omega)$, $g \in H^1(\Omega)$, γ is the boundary trace operator, and $\Omega \subset \mathbb{R}^N$ an open bounded Lipschitz set. A function $u \in H^1(\Omega)$ is a weak solution to (7.2.17) if

$$\langle Lu, \varphi \rangle = \langle f, \varphi \rangle, \quad \forall \varphi \in H_0^1(\Omega).$$

If we look for u as $u = v + g$, then

$$v \in H_0^1(\Omega), \quad \langle Lv, \varphi\rangle = \langle f - Lg, \varphi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \ \ \forall\varphi \in H_0^1(\Omega). \tag{7.2.18}$$

We can apply Theorem 7.2.12 to (7.2.18), and the conclusions of this theorem hold for (7.2.18) with $f - Lg$ instead of f.

Example 7.2.14 *Let us consider again the problem (4.4.3). It does not have a classical solution because the boundary data is not continuous. However, it has a unique $H^1(\Omega)$ solution. Indeed, if $G(x) = \mathbb{1}_{\{|x|<1\}} \ln|1 - \ln|x||$ then $u = G$ on $\partial\Omega$. As from Remark 6.3.5, $\ln|1 - \ln|x|| \in H^1(\Omega)$, from Remark 7.2.13 the problem (4.4.3) has a unique solution in $H^1(\Omega)$.*

7.2.3 Neumann problem

Using the method we developed in this chapter, we can solve the following "*Neumann problem*", which is similar to (7.1.1) but with the boundary condition (7.1.1b) replaced by the normal derivative of u, commonly referred to as the "*Neumann boundary condition*":

$$-\sum_{i,j=1}^{N} \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^{N} b_i\partial_i u + cu = f \in (H^1(\Omega))', \tag{7.2.19a}$$

$$\sum_{j=1,N} a_{ij}\partial_i u\nu_j = 0 \ \ \text{on} \ \ \partial\Omega, \tag{7.2.19b}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded Lipschitz set and $\nu = (\nu_1, \ldots, \nu_N)$ is the exterior unit normal vector to $\partial\Omega$. Note that by assuming u is smooth and (a_{ij}) is the identity matrix, the condition (7.2.19b) becomes $\nabla u \cdot \nu = 0$.

The weak form equation of (7.2.19) is given by

$$B(u, v) = \langle f, v\rangle_{(H^1(\Omega))'\times H^1(\Omega)}, \quad v \in H^1(\Omega), \tag{7.2.20}$$

which formally is obtained by multiplying (7.2.19) by $v \in H^1(\Omega)$, integrating by parts in Ω, and canceling the boundary term by using the boundary condition (7.2.19b).

As a consequence of Lax-Milgram lemma, we have the following theorem.

Theorem 7.2.15 *Let $f \in (H^1(\Omega))'$, $b = 0$, and $c \geq c_0 > 0$. Then the problem (7.2.20) has a unique solution $u \in H^1(\Omega)$. Furthermore, there exists a constant $C > 0$ independent of f such that*

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}.$$

Similar results to those of Theorem 7.2.12 for the Dirichlet problem (7.1.4) hold for the problem (7.2.20). We leave the details as an exercise. For more on this topic, see, for example, [5, 20].

Remark 7.2.16 *Let us see in what sense a weak solution to* (7.2.20) *satisfies the boundary condition* (7.2.19b)*. For simplicity, we assume $f \in L^2(\Omega)$ (see Problem 7.8 for $f \in (H^1(\Omega))'$). Then taking $v \in \mathcal{D}(\Omega)$ in* (7.2.20) *gives $T := \sum_{i,j=1}^N \partial_j(a_{ij}\partial_i u) = b \cdot \nabla u + cu - f$ in $\mathcal{D}'(\Omega)$. As $b \cdot \nabla u + cu - f \in L^2(\Omega)$, it follows $T = b \cdot \nabla u + cu$ in $L^2(\Omega)$. Then from* (7.2.20) *with $v \in H^1(\Omega)$, we get*

$$\langle T, v\rangle = \langle b \cdot \nabla u + cu - f, v\rangle = \int_\Omega (b \cdot \nabla u - cu)v dx - \langle f, v\rangle = -\int_\Omega \sum_{i,j=1}^N a_{ij}\partial_i u \partial_j v dx,$$

or equivalently

$$\left\langle \sum_{i,j=1}^N \partial_j(a_{ij}\partial_i u), v \right\rangle + \int_\Omega \sum_{i,j=1}^N a_{ij}\partial_i u \partial_j v dx = 0, \quad \forall v \in H^1(\Omega). \tag{7.2.21}$$

The equation (7.2.21) *defines how $\sum_{i,j=1}^N a_{ij}\partial_i u \nu_j = 0$ is understood. This makes sense because if $u \in C^2(\overline{\Omega})$ integrating by parts in* (7.2.21) *implies*

$$0 = \int_{\partial\Omega} \sum_{i,j=1}^N a_{ij}\partial_i u \nu_j v dx, \ \ \forall v \in H^1(\Omega),$$

which implies $\sum_{i,j=1}^N a_{ij}\partial_i u \nu_j = 0$.

7.3 Nonlinear second-order elliptic PDEs

Nonlinear PDEs describe a wide range of physical phenomena and are subject to intense research. There are several methods for solving nonlinear PDEs, such as variational, continuity, and fixed point methods. For a deeper analysis of these methods see, for example, [20].

In this section we give a short introduction to nonlinear second-order elliptic PDEs. We will consider only the fixed point method. Typically, this method consists of three steps: (i) write the solution to the nonlinear problem as a fixed point problem of a certain linear operator A, (ii) solve the linear problem associated with the operator A and establish appropriate estimates, and (iii) use the fixed point theorem.

In this short introduction to nonlinear PDEs, we will consider certain PDEs of the form

$$\sum_{i,j=1}^N -\partial_j(a_{i,j}(x,u)\partial_i u) + b(x,u) = f(x). \tag{7.3.1}$$

Let us cite the following theorem which asserts the existence of fixed points of a compact continuous operator; see, for example, [20].

Theorem 7.3.1 *Let $A : E \mapsto E$ be a continuous operator from the Banach space E to itself. Assume that there exists $K \subset E$, convex closed set, such that $A(K) \subset K$ and $A(K)$ is precompact*[2] *in E. Then A has a fixed point, i.e. there exists $u \in E$ such that $u = Au$.*

Now we give two examples, which demonstrate the use of the fixed point method to solve various problems similar to (7.3.1).

Example 7.3.2 *Consider the following nonlinear boundary value problem*

$$-u'' + h(u)u = f \quad \text{in } \Omega = (0,1), \quad u = 0 \ \text{ on } \partial\Omega, \tag{7.3.2}$$

with $f \in H^{-1}(\Omega)$, $0 \leq h \in C_b^0(\mathbb{R})$. Assuming a smooth solution u exists, multiplying (7.3.2) by $\varphi \in H_0^1(\Omega)$ and integrating in Ω yields

$$\int_\Omega u'\varphi' + h(u)u\varphi = \langle f, \varphi\rangle, \ \ \forall \varphi \in H_0^1(\Omega). \tag{7.3.3}$$

This is the weak form equation of (7.3.2), and we look for a solution $u \in H_0^1(\Omega)$ to it. For this we consider a fixed point approach as follows. Let $A : C^0(\overline{\Omega}) \mapsto C^0(\overline{\Omega})$, $Av = u$, where u is the solution to

$$\int_\Omega u'\varphi' + h(v)u\varphi = \langle f, \varphi\rangle, \ \ \forall \varphi \in H_0^1(\Omega). \tag{7.3.4}$$

If u is a fixed point of A, i.e. $u = Au$, then $u \in H_0^1(\Omega)$ and solves (7.3.3).
Claim: A is well-defined. *Indeed, for every $v \in C^0(\overline{\Omega})$ the equation (7.3.4) has a unique solution $u \in H_0^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$; see Theorem 7.2.8 and Theorem 6.3.9. Furthermore, the following estimate holds:*

$$\|Av\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}. \tag{7.3.5}$$

Claim: A is continuous in $C^0(\overline{\Omega})$. *Indeed, let $v_0, v \in C^0(\overline{\Omega})$ with v tending to v_0 in $C^0(\overline{\Omega})$, and set $u = Av$, $u_0 = Av_0$. Then from (7.3.4) with $\varphi = u - u_0$, we get*

$$\int_\Omega |u' - u_0'|^2 + h(v)(u-u_0)^2 + (h(v) - h(v_0))u_0(u-u_0) = 0,$$

which implies

$$|Av - Av_0|^2_{H^1(\Omega)} \ \ \leq \ \ \int_\Omega |h(v) - h(v_0)||u_0||Av - Av_0| \qquad (use\,\text{Hölder})$$

[2] A set $K \subset E$, E normed space, is precompact if its closure is compact.

$$\leq \|h(v) - h(v_0)\|_{C^0(\overline{\Omega})}\|u_0\|_{L^2(\Omega)}\|Av - Av_0\|_{L^2(\Omega)}.$$

This inequality in combination with Poincaré inequality yields

$$\|Av - Av_0\|_{H^1(\Omega)} \leq C\|h(v) - h(v_0)\|_{C^0(\overline{\Omega})}\|u_0\|_{L^2(\Omega)}.$$

From $h(v) \to h(v_0)$ *in* $C^0(\overline{\Omega})$ *and* $H^1_0(\Omega) \hookrightarrow C^0(\overline{\Omega})$*, we get* $\|Av - Av_0\|_{C^0(\overline{\Omega})} \to 0$ *as* $v \to v_0$ *in* $C^0(\overline{\Omega})$*, hence* A *is continuous from* $C^0(\overline{\Omega})$ *to itself.*
Claim: Let $K = H^1_0(\Omega)$. Then $A(K) \subset K$ and $A(K)$ is precompact in $C^0(\overline{\Omega})$. *This follows from the estimate (7.3.5) and* $H^1(\Omega) \overset{c}{\hookrightarrow} C^0(\overline{\Omega})$*; see Theorem 6.3.9.*
Claim: A has a fixed point. *Conditions of Theorem 7.3.1 are satisfied, so* A *has a fixed point* $u = Au \in H^1_0(\Omega)$*, which solves* (7.3.3).

Example 7.3.3 *Consider the following nonlinear PDE:*

$$-\Delta u + h(u)u = f \quad \text{in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{7.3.6}$$

with $\Omega \subset \mathbb{R}^2$ *an open bounded Lipschitz set,* $f \in H^{-1}(\Omega)$*,* $0 \leq h \in C^1_b(\mathbb{R})$*. Proceeding as in example above, we get this weak form equation*

$$u \in H^1_0(\Omega) \ \text{ such that } \int_\Omega \nabla u \cdot \nabla\varphi + h(u)u\varphi = \langle f, \varphi\rangle, \ \ \forall\varphi \in H^1_0(\Omega). \tag{7.3.7}$$

Similar to the method in Example 7.3.2, we consider a fixed point approach as follows. Let $A : L^2(\Omega) \mapsto L^2(\Omega)$*,* $Av = u$*, where* u *is the solution to*

$$\int_\Omega \nabla u \cdot \nabla\varphi + h(v)u\varphi = \langle f, \varphi\rangle, \ \ \forall\varphi \in H^1_0(\Omega). \tag{7.3.8}$$

Claim: A is well-defined. *Indeed, for* $v \in L^2(\Omega)$ *the equation (7.3.8) has a unique solution, see Theorem 7.2.8, and the following estimate holds:*

$$\|Av\|_{H^1_0(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}. \tag{7.3.9}$$

Claim: *A is continuous in* $L^2(\Omega)$*. Indeed, let* $v_0, v \in L^2(\Omega)$ *with* v *tending to* v_0 *in* $L^2(\Omega)$*, and set* $u = Av$*,* $u_0 = Av_0$*. Then, like in Example 7.3.2, from (7.3.8) with* $\varphi = u - u_0$ *we get*

$$\begin{aligned}
|Av - Av_0|^2_{H^1_0(\Omega)} &\leq \int_\Omega |h(v) - h(v_0)||u_0||Av - Av_0| \qquad (\textit{use Hölder}) \\
&\leq \|h(v) - h(v_0)\|_{L^2(\Omega)}\|u_0\|_{L^4(\Omega)}\|Av - Av_0\|_{L^4(\Omega)} \\
&\leq C\|h(v) - h(v_0)\|_{L^2(\Omega)}\|u_0\|_{L^4(\Omega)}\|Av - Av_0\|_{H^1_0(\Omega)},
\end{aligned}$$

where we used the fact that $H^1(\Omega) \hookrightarrow L^q(\Omega)$ *for all* $q \in [1, \infty)$*; see Remark 6.3.11. Therefore, using Poincare inequality* (6.5.1) *and* $|h(v) - h(v_0)| \leq \|h\|_{C^1_b(\mathbb{R})}|v - v_0|$ *implies*

$$\|Av - Av_0\|_{H^1_0(\Omega)} \leq C\|v - v_0\|_{L^2(\Omega)}\|u_0\|_{L^4(\Omega)},$$

which proves that A is continuous from $L^2(\Omega)$ to itself.
Claim: Let $K = H^1(\Omega)$. Then we have $A(K) \subset K$ and $A(K)$ is precompact in $L^2(\Omega)$. *This follows from the estimate (7.3.9) and the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$; see Theorem 6.3.10.*
Claim: A has a fixed point. *The conditions of Theorem 7.3.1 are satisfied. Then A has a fixed point $u = Au \in H_0^1(\Omega)$, solution to (7.3.6).*

Problems

Problem 7.1 *Prove that for every $\ell \in H^{-k}(\Omega)$, $k = 1$, there exists a unique $u \in H_0^k(\Omega)$ such that*

$$\sum_{|\alpha| \le k} (-1)^{|\alpha|} D^{2\alpha} u = \ell \ \ in \ \ \mathscr{D}'(\Omega).$$

Extend the result in the case $k \in \mathbb{N}$.

Problem 7.2 *Let $\Omega = (0,1)$, $c \in L^\infty(\Omega)$, and $f \in H^{-1}(\Omega)$. Consider the problem*

$$-u'' + c(x)u = f \quad in \ \Omega, \qquad u(0) = u(1) = 0.$$

Prove that it has a unique weak solution in $H_0^1(\Omega)$ if $\|c^-\|_{L^\infty(\Omega)} < 2$ (in fact, even if $\|c^-\|_{L^\infty(\Omega)} < \pi^2$).

Problem 7.3 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and consider the problem*

$$-\Delta u + \sum_{i=1,N} b_i \partial_i u + cu = f \quad in \ \Omega, \qquad u = 0 \quad on \ \partial\Omega,$$

with $b_i \in W^{1,\infty}(\Omega)$, $c \in L^\infty(\Omega)$, and $2c - \sum_{i=1,N} \partial_i b_i \ge 0$. Prove that there exists a unique weak solution $u \in H_0^1(\Omega)$.

Problem 7.4 *Use these inequalities*

$$\begin{aligned} \left\| \frac{u}{|x|} \right\|_{L^2(\mathbb{R}^N)} &\le \frac{2}{N-2} \|\nabla u\|_{L^2(\mathbb{R}^N)}, && \forall u \in H^1(\mathbb{R}^N), \ N \ge 3, \\ \left\| \frac{u}{|x| \ln(|x|/R)} \right\|_{L^2(\Omega)} &\le C \|\nabla u\|_{L^2(\Omega)}, && \forall u \in H_0^1(\Omega), \ \Omega \subset B(0,R) \subset \mathbb{R}^2 \end{aligned}$$

(see [51]) to show that the weak solutions to the following problems

$$-\Delta u = 1, \quad u \in H_0^1(B(0,R)\backslash\{0\}); \qquad -\Delta u = 1, \quad u \in H_0^1(B(0,R))$$

are the same.

Problem 7.5 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set, ν the unitary exterior normal vector on $\partial\Omega$ and consider the problem*

$$-\Delta u + u = f \in (H^1(\Omega)' \quad in\ \Omega, \qquad \partial_\nu u = g \quad on\ \partial\Omega,$$

where $g \in H^{-1/2}(\partial\Omega)$. Write the weak form of this problem and prove that this problem has a unique $H^1(\Omega)$ weak solution.

Problem 7.6 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set and ν the unitary exterior normal vector on $\partial\Omega$. Furthermore let $M_0^1(\Omega) = \left\{u \in H^1(\Omega),\ \int_\Omega u dx = 0\right\}$ and consider the problem*

$$\begin{aligned} -\Delta u &= f \quad in\ \Omega, \quad f \in (H^1(\Omega))', \ \langle f, 1\rangle = 0, \\ \partial_\nu u &= 0 \quad on\ \partial\Omega. \end{aligned}$$

i) Write the weak form of this problem.

ii) Prove that Poincaré inequality holds in $M_0^1(\Omega)$.

iii) Prove that this problem has a unique weak solution in $M_0^1(\Omega)$.

Problem 7.7 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected Lipschitz set and γ be a relatively open set on $\partial\Omega$ with nonzero $\mathcal{H}^{N-1}$ measure. Furthermore let $H_\gamma^1(\Omega)$ be the closure for the $H^1(\Omega)$ norm of $\left\{v \in C^1(\overline{\Omega}),\quad v|_\gamma = 0\right\}$, and consider the problem: $u \in H_\gamma^1(\Omega)$ solution to*

$$-\Delta u = f \quad in\ \Omega, \qquad u = 0 \quad on\ \gamma, \quad \partial_\nu u = 0 \quad on\ \partial\Omega\backslash\gamma, \quad f \in L^2(\Omega).$$

i) Write the weak form of this problem.

ii) Prove that Poincaré inequality holds in $H_\gamma^1(\Omega)$.

iii) Prove that this problem has a unique weak solution in $H_\gamma^1(\Omega)$.

Problem 7.8 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set and let u be a solution to (7.2.20), with $b = 0$ and $c \geq c_0 > 0$. Show that $L^2(\Omega) \overset{d}{\hookrightarrow} (H^1(\Omega))'$ and, by approximating $f \in (H^1(\Omega))'$ by a sequence (f_n) in $L^2(\Omega)$, give a meaning to (7.2.21) in the case $f \in (H^1(\Omega))'$.*

Problem 7.9 *Consider the problem*

$$\begin{aligned} -\Delta^2 u + c(x)u &= f \quad in\ \Omega \subset \mathbb{R}^N, \qquad where\ \ \Delta^2 u = \Delta(\Delta(u)), \\ u = \partial_\nu u &= 0 \quad on\ \partial\Omega, \end{aligned}$$

with Ω an open bounded Lipschitz set, ν the exterior unitary normal on $\partial\Omega$, and $0 < c \in C^0(\overline{\Omega})$. Write the weak form of this problem and use Corollary 9.5.4 to prove that it has a unique weak solution $u \in H_0^2(\Omega)$ for every $f \in H^{-2}(\Omega)$.

Problem 7.10 *Let $\Omega \subset \mathbb{R}$ be an open bounded interval, $a \in C^0(\mathbb{R})$, $m \le a \le M$, with $m, M \in (0,\infty)$, $0 \le c \in L^\infty(\Omega)$.*

i) Write the weak form of this problem

$$-(a(u)u')' + c(x)u = f \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega).$$

ii) Prove that this problem has a weak solution for every $f \in H^{-1}(\Omega)$ (you may consider the map $A : C^0(\overline{\Omega}) \mapsto C^0(\overline{\Omega})$ with $u = Av$ the solution to $-(a(v)u')' + c(x)u = f$ in $H_0^1(\Omega)$ and use Theorem 7.3.1).

Problem 7.11 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set and consider the problem*

$$-\Delta u = e^{-u^2} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

i) Write the weak form of this problem.

ii) Prove that this problem has a unique weak solution (you may apply Theorem 7.3.1) to the map $A : L^q(\Omega) \mapsto L^q(\Omega)$, for a certain q, with $u = A(v)$ solving $-\Delta u = e^{-v^2}$ in $H_0^1(\Omega)$).

Problem 7.12 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set, $N \le 5$, and consider the problem:*

$$u \in H_0^1(\Omega), \quad -\Delta u + h(u)u = f \quad \text{in } \Omega,$$

where $f \in H^{-1}(\Omega)$, $0 \le h \in C_b^1(\mathbb{R})$.

i) Write the weak form equation of this problem.

ii) Prove that this problem has a unique weak solution (you may apply Theorem 7.3.1) to the map $A : L^q(\Omega) \mapsto L^q(\Omega)$, for a certain q, with $u = A(v)$ solving $-\Delta u + h(v)u = f$ in $H_0^1(\Omega)$).

8. Second-order parabolic and hyperbolic PDEs

In this chapter we consider the evolution equation of the form

$$u'(t) + Lu(t) = f(t), \quad t \in (0,T), \tag{8.0.1a}$$

$$u(0) = u_0, \tag{8.0.1b}$$

where $T > 0$, $u : (0,T) \mapsto V$ is a map, V is a Banach space, $u' := \dfrac{du}{dt}$ is the time derivative of u, $L : V \mapsto V'$ is an operator, and u_0 and f are given. With the same notations as above, we will also consider the second-order evolution equation of the form

$$u''(t) + Lu(t) = f(t), \quad t \in (0,T), \tag{8.0.2a}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{8.0.2b}$$

where $u'' := \dfrac{d^2u}{dt^2}$ is the second-order time derivative of u, and u_0, u_1 are given. We will define in sections 8.3 and 8.4 in which sense (8.0.1) and (8.0.2) are understood.

Often, the equations (8.0.1) and (8.0.2) are referred to as heat equation and wave equation, respectively. In the simplest form they are given by

$$\left\{ \begin{array}{rcl} u'(t) - \Delta u(t) &=& f(t), \\ u(0) &=& u_0, \end{array} \right. \quad \textit{resp.} \quad \left\{ \begin{array}{rcll} u''(t) - \Delta u(t) &=& f(t), & t \in (0,T), \\ u(0) = u_0, \quad u'(0) &=& u_1. & \end{array} \right.$$

There are different methods to analyze the equations of type (8.0.1) and (8.0.2), such as Hilbert spaces, variational (or weak formulation), semigroup, Laplace transform, and spectral decomposition methods; see, for example, [5, 11, 27, 35, 53]. We will only present in detail the method of spectral decomposition.

8.1 Heat and wave equations and the method of separation of variables

Before we develop the analysis of (8.0.1) and (8.0.2), we will present the method of separation of variables for heat and wave equations in dimension one with $L = -\Delta$. It

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6_8

serves as a motivation for the method of separation of variables in the abstract case that we will consider in this chapter.

8.1.1 Heat equation and the method of separation of variables

Here we consider the heat equation in one space dimension

$$u' - \Delta u = 0 \quad in \;\; \Omega \times (0,T) := (0,\pi) \times (0,T), \tag{8.1.1a}$$
$$u = 0 \quad on \;\; \partial\Omega \times (0,T), \tag{8.1.1b}$$
$$u = g \quad on \;\; \Omega \times \{0\}. \tag{8.1.1c}$$

Here $u = u(x,t)$, $u(\cdot,t) : \Omega \mapsto \mathbb{R}$, u' is the derivative of u w.r.t. t, and $\Delta u = \partial^2_{xx} u$.

Without developing a functional framework, at this point it is not clear what kind of solution to look for (8.1.1). However, we will follow the well-known method of "*separation of variables*" for non-rigorously solving (8.1.1). At the beginning of the next section, we will provide more comments about this method before we formalize it for more general problems.

We look for $u = u(x,t)$ in the form $u = V(t)W(x)$. Replacing u in (8.1.1a) gives

$$V'(t)W(x) - V(t)W''(x) = 0.$$

Assuming $VW \neq 0$ for all (x,t), from the last equation we obtain $\frac{V'(t)}{V(t)} = \frac{W''(x)}{W(x)}$, which implies $\frac{V'(t)}{V(t)} = \lambda$, $\frac{W''(x)}{W(x)} = \lambda$, for a certain constant$\lambda$, or equivalently

$$V' - \lambda V = 0, \quad W'' - \lambda W = 0. \tag{8.1.2}$$

The equations (8.1.2) are respectively of first and second order, linear, and homogeneous, so their solution spaces are respectively of dimension one and two. Their general solutions are

$$V(t) = \gamma e^{\lambda t}, \quad W(x) = \begin{cases} \alpha x + \beta, & \lambda = 0, \\ \alpha e^{x\sqrt{\lambda}} + \beta e^{-x\sqrt{\lambda}}, & \lambda \neq 0, \end{cases} \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

Therefore

$$u(x,t) = \begin{cases} e^{\lambda t}(Ax + B), & \lambda = 0, \\ e^{\lambda t}(Ae^{x\sqrt{\lambda}} + Be^{-x\sqrt{\lambda}}), & \lambda \neq 0, \end{cases} \quad A, B \in \mathbb{R}. \tag{8.1.3}$$

Such a u solves (8.1.1a) for arbitrary A, B, and λ. The constants A, B, and λ are chosen such that u solves (8.1.1b) and (8.1.1c). Let us first deal with (8.1.1b). As we must have $u(0,t) = u(\pi,t) = 0$ for all $t \in (0,T)$, in the case $\lambda = 0$ we get $B = A\pi + B = 0$, so $A = B = 0$ and $u = 0$. In the case $\lambda \neq 0$ it follows $A + B = Ae^{\pi\sqrt{\lambda}} + Be^{-\pi\sqrt{\lambda}} = 0$. Hence

$$B = -A, \quad e^{2\pi\sqrt{\lambda}} = 1, \quad \sqrt{\lambda} = ik, \quad \lambda = -k^2, \;\; k \in \mathbb{N}.$$

Therefore, all non-trivial $u(x,t)$ given by (8.1.3) have the form

$$u_k(x,t) \;=\; Ae^{-tk^2}(e^{ikx} - e^{-ikx}) = C_k e^{-tk^2} \sin(kx)\,, \quad k \in \mathbb{N}, \quad C_k \in \mathbb{R}.$$

From the linearity of (8.1.1a) and (8.1.1b), it follows (not rigorously) that

$$u(x,t) = \sum_{k\in\mathbb{N}} u_k(x,t) = \sum_{k\in\mathbb{N}} C_k e^{-tk^2} \sin(kx) \tag{8.1.4}$$

solves (8.1.1a) and (8.1.1b). The coefficients C_k are unknown, but we can define them by imposing the condition (8.1.1c). Replacing u of (8.1.4) in (8.1.1c) implies

$$u(x,0) = \sum_{k\in\mathbb{N}} C_k \sin(kx) = g(x), \quad x \in (0,\ell). \tag{8.1.5}$$

Using the relations (4.1.10) gives $C_k = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin(ks)\, g(s)ds$, which implies

$$u(x,t) = \frac{2}{\pi}\sum_{k\in\mathbb{N}} e^{-tk^2}\sin(kx)\left(\int_0^\pi \sin(ks)\, g(s)ds\right). \tag{8.1.6}$$

We note that (8.1.6) gives a candidate for a solution to (8.1.1). Ideally, one looks for a "*classical solution*" to (8.1.1), which is $u \in C^2(\Omega\times(0,T)) \cap C^0(\overline{\Omega}\times[0,T))$ satisfying (8.1.1) pointwise. Classical solutions require strong assumptions on the data f, u_0, which narrow the class of f and u_0. In this chapter we look for weak solutions, which exist for a large range of data; see section 8.3.

Remark 8.1.1 *Using the same method as above, we can also solve the problem*

$$\begin{aligned} u' - c^2\Delta u &= 0 \quad \text{in } \Omega\times(0,T) := (0,\ell)\times(0,T), && \text{(8.1.7a)}\\ u &= 0 \quad \text{on } \partial\Omega\times(0,T), && \text{(8.1.7b)}\\ u &= p \quad \text{on } \Omega\times\{0\}. && \text{(8.1.7c)} \end{aligned}$$

Indeed, we can change the variables as follows:

$$y = \frac{\pi}{\ell}x, \quad \tau = \left(c\frac{\pi}{\ell}\right)^2 t, \qquad v(y,\tau) = u(x,t).$$

If furthermore $g(y) = p(x)$ then $v(x,\tau)$ solves

$$\begin{aligned} v' - \Delta v &= 0 \quad \text{in } (0,\pi)\times\left(0,\left(c\frac{\pi}{\ell}\right)^2 T\right), && \text{(8.1.8a)}\\ v &= 0 \quad \text{on } \{0,\pi\}\times\left(0,\left(c\frac{\pi}{\ell}\right)^2 T\right), && \text{(8.1.8b)}\\ v &= g \quad \text{on } (0,\pi)\times\{0\}, && \text{(8.1.8c)} \end{aligned}$$

and is given by (8.1.6). Then

$$u(x,t) = \frac{2}{\ell}\sum_{k\in\mathbb{N}} e^{-t\left(ck\frac{\pi}{\ell}\right)^2}\sin\left(k\frac{\pi}{\ell}x\right)\left(\int_0^\ell \sin\left(k\frac{\pi}{\ell}s\right)p(s)ds\right). \tag{8.1.9}$$

The method of separation of variables can be used to solve (8.1.1) with different boundary conditions than (8.1.1b), for example $u_x(0,t) = 0$ and $u(\ell,t) + u_x(\ell,t) = 0$. The method is the same as above, except that after applying these conditions we get different formulas for the coefficients A, B, λ, and C_k.

Example 8.1.2 *Let us find a classical solution to the problem*

$$\begin{aligned} u_t - 16u_{xx} &= 0 \ \text{ in } \ (0,\pi)\times(0,\infty), && (8.1.10a)\\ u(0,t) = u(\pi,t) &= 0 \ \text{ for } \ t>0, && (8.1.10b)\\ u(x,0) &= x(\pi-x) \ \text{ for } \ x\in(0,\pi), && (8.1.10c)\end{aligned}$$

by using the method of separation of variables. We can use (8.1.9) with $\ell = \pi$, $g(x) = x(\pi - x)$, and $c = 4$. Integrating by parts

$$\int_0^\ell \sin\left(k\frac{\pi}{\ell}s\right) g(s)ds = \int_0^\pi \sin(ks)\, s(\pi-s)ds = 2\frac{1-(-1)^k}{k^3}$$

gives

$$u(x,t) \ = \ \frac{4}{\pi}\sum_{k\in\mathbb{N}} \frac{1-(-1)^k}{k^3} e^{-t(4k)^2}\sin(kx) = \frac{4}{\pi}\sum_{k\in\mathbb{N}} u_k(x,t). \tag{8.1.11}$$

We show that u is a classical solution to (8.1.10). Indeed, note first that the series (8.1.11) converges in $C^0([0,\pi]\times[0,T])$, for arbitrary $T>0$ because $|u_k(x,t)| \le \frac{2}{k^3}$ for all $(x,t)\in[0,\pi]\times[0,\infty)$. Then the claim follows from Weierstrass M-test. Therefore, $u\in C^0([0,\pi]\times[0,\infty))$ and u satisfies (8.1.10b), (8.1.10c). As $|\partial_x u_k| + |\partial_t u_k| \le \frac{C}{k^2}$ for all $(x,t)\in[0,\pi]\times[0,\infty))$, by using again Weierstrass M-test we get $u\in C^1([0,\pi]\times[0,\infty))$.

Furthermore, $u\in C^2((0,\pi)\times(0,\infty))$ and satisfies (8.1.10a). Indeed, all u_k satisfy (8.1.10a). Next we point out that given $\epsilon > 0$ and $i,j\in\{x,t\}$ we have s

$$|\partial_{i,j}u_k| \ \le \ C\frac{e^{-\epsilon(4k)^2}}{k} \quad \text{in } \ [0,\pi]\times[\epsilon,T], \quad \text{and} \quad \sum_{k\in\mathbb{N}}\frac{e^{-\epsilon(4k)^2}}{k} < \infty.$$

Then from Weierstrass M-test the series (8.1.11) converges in $C^2([0,\pi]\times[\epsilon,T])$ and the claim follows from the arbitrariness of ϵ and T. Finally, we point out that $\lim_{t\to\infty} u(x,t) = 0$ because

$$\lim_{t\to\infty}|u(x,t)| \le C\left(\sum_{k\in\mathbb{N}}\frac{1}{k^3}\right)\lim_{t\to\infty} e^{-16t} = 0.$$

8.1.2 Wave equation and the method of separation of variables

Now we use the method of separation of variables again, this time to solve the wave equation in dimension one, namely

$$u_{tt} - u_{xx} = 0 \quad in \;\; \Omega \times (0,T), \quad \Omega = (0,\pi), \;\; T > 0 \tag{8.1.12a}$$
$$u = 0 \quad on \;\; \partial\Omega \times (0,T), \tag{8.1.12b}$$
$$u = g, \; u_t = h \quad on \;\; \Omega \times \{0\}. \tag{8.1.12c}$$

Disregarding the issues of existence and uniqueness, we are interested only in finding a formula for the solution u. We assume that $u = u(x,t)$ can be written as $u(x,t) = V(t)W(x)$. Replacing u in (8.1.12a) gives

$$V''(t)W(x) - V(t)W''(x) = 0.$$

Assuming $VW \neq 0$, from the last equation we obtain $\frac{V''(t)}{V(t)} = \frac{W''(x)}{W(x)}$. Then, necessarily there exists a constant λ such that $\frac{V''(t)}{V(t)} = \lambda, \quad \frac{W''(x)}{W(x)} = \lambda$, or equivalently

$$V'' - \lambda V = 0, \quad W'' - \lambda W = 0. \tag{8.1.13}$$

The equations (8.1.13) are of second order, linear, and homogeneous, so their solution spaces are of dimension two. The general solutions are of the form

$$V(t) = \begin{cases} At + B, & \lambda = 0, \\ Ae^{t\sqrt{\lambda}} + Be^{-t\sqrt{\lambda}}, & \lambda, \neq 0, \end{cases} \qquad W(x) = \begin{cases} Cx + D, & \lambda = 0, \\ Ce^{x\sqrt{\lambda}} + De^{-x\sqrt{\lambda}}, & \lambda \neq 0, \end{cases}$$

with A, B, C, D being arbitrary constants. Therefore

$$u(x,t) = \begin{cases} (At + B)(Cx + D), & \lambda = 0, \\ (Ae^{t\sqrt{\lambda}} + Be^{-t\sqrt{\lambda}})(Ce^{x\sqrt{\lambda}} + De^{-x\sqrt{\lambda}}), & \lambda \neq 0, \end{cases} \tag{8.1.14}$$

is always a solution to (8.1.12a). The constants A, B, C, D, and λ are chosen such that u solves (8.1.12b), (8.1.12c). Let us first deal with (8.1.12b). The conditions $u(0,t) = u(\pi,t) = 0$, $t \in (0,T)$ are equivalent to

$$\begin{array}{ll} D = C\pi + D = 0, & \lambda = 0, \\ C + D = Ce^{\pi\sqrt{\lambda}} + De^{-\pi\sqrt{\lambda}} = 0, & \lambda \neq 0. \end{array}$$

Then necessarily

$$\begin{array}{ll} C = D = 0, & \lambda = 0, \\ D = -C, \quad e^{2\pi\sqrt{\lambda}} = 1, \quad \sqrt{\lambda} = ik, \quad \lambda = -k^2, \quad k \in \mathbb{N}, & \lambda \neq 0. \end{array}$$

Therefore, $u(x,t)$ given by (8.1.14) looks like

$$u_k(x,t) \;\; = \;\; C(Ae^{ikt} + Be^{-ikt})(e^{ikx} - e^{-ikx})$$

$$\begin{aligned} &= 2iC((A+B)\cos(kt) + i(A-B)\sin(kt))\sin(kx) \\ &=: (A_k\cos(kt) + B_k\sin(kt))\sin(kx). \end{aligned}$$

It follows that

$$u(x,t) = \sum_{k\in\mathbb{N}} u_k(x,t) = \sum_{k\in\mathbb{N}} (A_k\cos(kt) + B_k\sin(kt))\sin(kx) \tag{8.1.15}$$

solves (8.1.12a) and (8.1.12b). Now, it remains to find A_k and B_k. They are chosen such that u given by (8.1.15) satisfies (8.1.12c), which implies

$$u(x,0) = \sum_{k\in\mathbb{N}} A_k\sin(kx) = g(x), \quad u_t(x,0) = \sum_{k\in\mathbb{N}} B_k k\sin(kx) = h(x). \tag{8.1.16}$$

If the functions g and h have the following sine Fourier expansions

$$g(x) = \sum_{k\in\mathbb{N}} g_k\sin(kx), \quad h(x) = \sum_{k\in\mathbb{N}} h_k\sin(kx), \quad x\in(0,\pi),$$

then from (8.1.16) we obtain $A_k = h_k$, $B_k = \dfrac{g_k}{k}$ and therefore

$$u(x,t) = \sum_{k\in\mathbb{N}} \left(g_k\cos(kt) + \frac{h_k}{k}\sin(kt) \right)\sin(kx) \tag{8.1.17}$$

solves (8.1.12).

Like for (8.1.6), (8.1.17) gives a candidate for the solution to (8.1.12). On the first attempt, we look for a "*classical solution*" to (8.1.12), which is $u \in C^2(\Omega\times(0,T)) \cap C^1(\Omega\times[0,T)) \cap C^0(\overline{\Omega}\times[0,T))$ satisfying (8.1.12) pointwise. Classical solutions require strong assumptions on the data f, u_0, u_1. We will study the existence of weak solutions, which exist for a large range of data f, u_0, and u_1; see section 8.4.

Remark 8.1.3 *Using the same method we can solve the problem*

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \quad \text{in } \Omega\times(0,T) := (0,\ell)\times(0,T), && (8.1.18)\\ u &= 0 \quad \text{on } \partial\Omega\times(0,T), && (8.1.19)\\ u = p,\ u_t &= q \quad \text{on } \Omega\times\{0\}. && (8.1.20) \end{aligned}$$

Indeed, we can change the variables as follows:

$$y = \frac{\pi}{\ell}x, \quad \tau = c\frac{\pi}{\ell}t, \qquad v(y,\tau) = u(x,t).$$

Note that we have

$$u_{tt}(x,t) = \left(c\frac{\pi}{\ell}\right)^2 v_{\tau\tau}(y,\tau), \quad c^2 u_{xx}(x,t) = \left(c\frac{\pi}{\ell}\right)^2 v_{yy}(y,t).$$

Therefore, if $g(y) = p(x)$, $h(y) = q(x)$ *then* $v(y,t)$ *solves*

$$\begin{aligned} v_{\tau\tau} - v_{yy} &= 0 \quad \text{in } (0,\pi) \times \left(0, c\frac{\pi}{\ell}T\right), && (8.1.21)\\ v &= 0 \quad \text{on } \{0,\pi\} \times \left(0, c\frac{\pi}{\ell}T\right), && (8.1.22)\\ v = g,\ v_\tau &= c\frac{\pi}{\ell}h \quad \text{on } (0,\pi) \times \{0\}. && (8.1.23) \end{aligned}$$

Note that the sine Fourier coefficients g_k*, resp.* h_k*, of* g*, resp.* h*, are related to the sine Fourier coefficients* p_k*, resp.* q_k*, of* p*, resp.* q*, by resp.* $g_k = p_k$*,* $h_k = q_k$*. Then from (8.1.17) and* $u(x,t) = v\left(\frac{\pi}{\ell}x, c\frac{\pi}{\ell}t\right)$*, we get*

$$u(x,t) = \sum_{k\in\mathbb{N}} \left(p_k \cos\left(ck\frac{\pi}{\ell}t\right) + \frac{\ell}{c\pi}\frac{q_k}{k}\sin\left(ck\frac{\pi}{\ell}t\right)\right) \sin\left(k\frac{\pi}{\ell}x\right),$$

where p_k *and* q_k *are given by*

$$p_k = \frac{2}{\ell}\int_0^\ell \sin\left(k\frac{\pi}{\ell}x\right) p(x)dx, \quad q_k = \frac{2}{\ell}\int_0^\ell \sin\left(k\frac{\pi}{\ell}x\right) q(x)dx.$$

Finally we point out that following the same method, we can consider more general boundary conditions than (8.1.12b), such as

$$a_0 u(0,t) + b_0 u_x(0,t) = 0, \quad a_\pi u(\pi,t) + b_\pi u_x(\pi,t) = 0, \tag{8.1.24}$$

with a_0, b_0, a_π, b_π given constants, $a_0^2 + b_0^2 > 0$, $a_\pi^2 + b_\pi^2 > 0$.

D'Alembert solution to the wave equation

When the space domain of the wave equation is $\mathbb{R}$, we can get a formula for the solution. Indeed, consider

$$\begin{aligned} u_{tt} - c^2 u_{xx} = 0 &\quad \text{in } \mathbb{R} \times (0,\infty), && (8.1.25a)\\ u = g,\ u_t = h &\quad \text{on } \mathbb{R} \times \{0\}. && (8.1.25b) \end{aligned}$$

As we have seen in Example 2.0.4, the general solution to (8.1.25a) is given by

$$u(x,t) = F(x-ct) + G(x+ct), \tag{8.1.26}$$

with F and G being two $C^2(\mathbb{R})$ functions. The functions $F(x-ct)$ and $G(x+ct)$ are called "*traveling wave solutions*"; more precisely $F(x-ct)$ is called "*forward wave*", while $G(x+ct)$ is called "*backward wave*". The curves $x \pm ct = x_0$, $x_0 \in \mathbb{R}$ are called "*characteristic curves*" of the wave equation.

To identify g and h, we replace (8.1.26) in (8.1.25b) and get

$$F(x) + G(x) = g(x), \qquad c(-F'(x) + G'(x)) = h(x).$$

It follows that

$$F(x) + G(x) = g(x), \qquad c(-F(x) + G(x)) = c(-F(0) + G(0)) + \int_0^x h(s)ds.$$

Hence

$$\begin{aligned} F(x) &= \frac{1}{2}g(x) + \frac{1}{2}(F(0) - G(0)) - \frac{1}{2c}\int_0^x h(s)ds, \\ G(x) &= \frac{1}{2}g(x) + \frac{1}{2}(-F(0) + G(0)) + \frac{1}{2c}\int_0^x h(s)ds, \quad \textit{and} \\ u(x,t) &= F(x - ct) + G(x + ct) \\ &= \frac{1}{2}(g(x - ct) + g(x + ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds. \end{aligned} \tag{8.1.27}$$

The solution given by the formula (8.1.27) is called the "*d'Alembert solution to the wave equation* (8.1.25)".

8.2 Some preliminary results

In general, the formulas (8.1.6) and (8.1.17) do not provide classical solutions. However, the heat and wave equations describe natural phenomena and expanding their solutions in series (8.1.6) and (8.1.17) is meaningful. This means that the concept of classical solutions is restrictive. Instead, the concept of weak solutions is more general and includes the classical solutions. Before we develop the analysis of the existence of weak solutions to evolution equations, we will present some preliminary results.

A careful view of the method of separation of variables for heat and wave equations shows that in all cases the solution is given in the form of a series like

$$u(t) := u(\cdot, t) = \sum_{n=1}^{\infty} c_n(t) \sin(n(\cdot)), \quad (\cdot) \in \Omega = (0, \pi),$$

with appropriate functions c_n. Hence, $u(\cdot, t)$ belongs to *span* $\{u_n = \sin(nx)),\ n \in \mathbb{N}\}$. Note that u_n and $\lambda_n = n^2$, $n = 1, 2, \ldots$, are the only eigenfunction/eigenvalue pairs of the operator $-\Delta : u \in H_0^1(\Omega) \cap H^2(\Omega) \mapsto L^2(\Omega)$. Note also that if T denotes the inverse of $-\Delta$, $T = (-\Delta)^{-1}$, then T is a compact self-adjoint operator, and u_n, n^{-2} are the only eigenfunction/eigenvalue pairs of the operator T.

The property of writing the solution as a series of eigenfunctions u_n of T, often called "*spectral decomposition*", holds in general for any self-adjoint (symmetric) compact operator T in a separable Hilbert space. It allows one to look for the solution to a certain PDE involving such an operator T as a series of eigenfunctions of T. We have this general result; see, for example, [5, 13, 29].

Theorem 8.2.1 (Spectral decomposition) *Let $(H, (\cdot,\cdot))$ be a separable Hilbert space and $T : H \mapsto H$ be a self-adjoint compact operator, i.e.*

$$\begin{aligned} \textit{[self-adjointness]} &: \quad (Tu, v) = (u, Tv) \text{ for every } u, v \in H, \text{ and} \qquad (8.2.1)\\ \textit{[compactness]} &: \quad \text{for every bounded sequence } (u_n) \text{ in } H, (T(u_n)) \text{ has} \\ &\quad\ \ \text{a convergent subsequence in } H. \qquad (8.2.2)\end{aligned}$$

Then there exists a countably infinite orthonormal basis (φ_n) of H consisting of eigenvectors of T, with corresponding eigenvalues $\{\mu_n\} \subset \mathbb{R}$ and $\lim_{n\to\infty} \mu_n = 0$.

We will restrict the analysis of (8.0.1) and (8.0.2) to the cases where L is given by

$$\begin{cases} L: \ H_0^1(\Omega) \ \mapsto \ H^{-1}(\Omega), \\ \qquad u \ \mapsto \ Lu = -\displaystyle\sum_{i,j=1}^{N} \partial_j(a_{ij}\partial_i u) + cu, \\ \qquad\qquad \langle Lu, \varphi\rangle = \displaystyle\int_\Omega \left(\sum_{i,j=1}^{N} a_{ij}\partial_i u \partial_j \varphi + cu\varphi\right) dx, \quad \forall \varphi \in H_0^1(\Omega). \end{cases} \tag{8.2.3}$$

Here and throughout this chapter, unless otherwise stated, $\Omega \subset \mathbb{R}^N$ is an open bounded set, and a_{ij} and c satisfy

$$A := (a_{ij}) = (a_{ji}) \in W^{1,\infty}(\Omega; \mathbb{R}^{N\times N}), \quad c \in L^\infty(\Omega), \tag{8.2.4}$$

$$\exists k, K > 0, \ \ \forall 0 \neq \xi \in \mathbb{R}^N, \ \ k|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(\cdot)\xi_i\xi_j \le K|\xi|^2, \ \ a.e. \text{ in } \Omega. \tag{8.2.5}$$

In what follows, we will use these notations:

$$\begin{aligned} (u, v) &:= (u, v)_{L^2(\Omega)}, & |u| &:= (u, u)^{1/2}, \\ ((u, v)) &:= (u, v)_{H_0^1(\Omega)} = \langle Lu, v\rangle, & \|u\| &:= ((u, u))^{1/2}, \end{aligned}$$

where for the equivalence of the norm in $H_0^1(\Omega)$ we have used the Poincaré inequality in $H_0^1(\Omega)$ and (8.2.5). The following theorem characterizes the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ in terms of series, involving the eigenvalues and eigenfunctions of L.

Theorem 8.2.2 *Set $T : L^2(\Omega) \mapsto L^2(\Omega)$, $Tf = u$, where $Lu = f$, $u \in H_0^1(\Omega)$.*

a) L is positive and invertible from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$, and T is a well-defined self-adjoint and compact operator.

b) There exists a countably infinite orthonormal basis $\{\varphi_n\}$ of $L^2(\Omega)$ for the inner product (u, v), consisting of eigenvectors $\{\varphi_n\}$ of L, $\langle L\varphi_n, \varphi\rangle = \lambda_n(\varphi_n, \varphi)$ for all $\varphi \in H_0^1(\Omega)$, with moreover $\{\lambda_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \lambda_n = \infty$, and

$$L^2(\Omega) \ = \ \left\{ u = \sum_{n=1}^{\infty} (u, \varphi_n)\varphi_n, \quad |u|^2 = \sum_{n=1}^{\infty} (u, \varphi_n)^2 < \infty \right\}. \tag{8.2.6}$$

c) Similarly, $\{\psi_n\} := \{\varphi_n/\sqrt{\lambda_n}\}$ *is an orthonormal basis of* $H_0^1(\Omega)$ *for the inner product* $((u,v))$ *and*

$$H_0^1(\Omega) = \left\{ u = \sum_{n=1}^{\infty}(u,\varphi_n)\varphi_n, \quad \|u\|^2 = \sum_{n=1}^{\infty}\lambda_n(u,\varphi_n)^2 < \infty \right\}. \tag{8.2.7}$$

d) Let

$$dom(L) = \left\{ u = \sum_{n=1}^{\infty}(u,\varphi_n)\varphi_n, \;\; |u|^2_{dom(L)} := \sum_{n=1}^{\infty}\lambda_n^2(u,\varphi_n)^2 < \infty \right\}.$$

Then $dom(L) \subset L^2(\Omega)$, $Lu = \sum_{n=1}^{\infty}\lambda_n(u,\varphi_n)\varphi_n \in L^2(\Omega)$ *and* $\|Lu\|_{L^2(\Omega)} = |u|_{dom(L)}$ *for every* $u \in dom(L)$. *Furthermore,* $L \in \mathcal{L}(dom(L); L^2(\Omega))$ *and* $dom(L)$ *equipped with the norm* $\|u\|_{dom(L)} = (\|u\|^2_{L^2(\Omega)} + |u|^2_{dom(L)})^{1/2}$ *is a Hilbert space.*

Proof. The claim a) follows from Theorem 7.2.8 and the compact embedding $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$. The claim b) follows from the classical theorem of spectral decomposition of compact self-adjoint operators; see Theorem 8.2.1. For c), clearly $\{\psi_n\}$ is orthonormal in $H_0^1(\Omega)$ because

$$((\psi_m,\psi_n)) = \langle L\psi_m,\psi_n\rangle = \lambda_m(\psi_m,\psi_n) = (\varphi_m,\varphi_n)\sqrt{\frac{\lambda_m}{\lambda_n}}.$$

Furthermore $\{\psi_n,\ n\in\mathbb{N}\}$ is dense in $H_0^1(\Omega)$, because if $u \in H_0^1(\Omega)$ and $((u,\psi_n)) = 0$ for all $n \in \mathbb{N}$ then

$$0 = ((u,\psi_n)) = \langle L\psi_n, u\rangle = \lambda_n(\psi_n,u) = \sqrt{\lambda_n}(\varphi_n,u)$$

which implies $u=0$ because $\{\varphi_n,\ n\in\mathbb{N}\}$ is dense in $L^2(\Omega)$. For (8.2.7) we note that if $u \in H_0^1(\Omega)$ then $u = \sum_{n=1}^{\infty}((u,\psi_n))\psi_n$ and

$$\|u\|^2 = \sum_{n=1}^{\infty}((u,\psi_n))^2 = \sum_{n=1}^{\infty}\langle L\psi_n,u\rangle^2 = \sum_{n=1}^{\infty}\lambda_n^2(\psi_n,u)^2 = \sum_{n=1}^{\infty}\lambda_n(u,\varphi_n)^2.$$

For d), let $u \in dom(L)$, so $u = \sum_{n=1}^{\infty}(u,\varphi_n)\varphi_n$ and $\sum_{n=1}^{\infty}\lambda_n^2(u,\varphi_n)^2 < \infty$. Clearly $u \in L^2(\Omega)$ because $\lambda_n \ge \lambda_1 > 0$. For arbitrary $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned}
\langle Lu,\varphi\rangle &= \langle u, L\varphi\rangle \\
&= \left\langle \sum_{n=1}^{\infty}(u,\varphi_n)\varphi_n, L\varphi \right\rangle = \sum_{n=1}^{\infty}(u,\varphi_n)\langle L\varphi_n,\varphi\rangle \\
&= \sum_{n=1}^{\infty}\lambda_n(u,\varphi_n)(\varphi_n,\varphi) = \left\langle \sum_{n=1}^{\infty}\lambda_n(u,\varphi_n)\varphi_n,\varphi \right\rangle,
\end{aligned}$$

which shows that $Lu = \sum_{n=1}^{\infty} \lambda_n (u, \varphi_n)\varphi_n$ and $\|Lu\|_{L^2(\Omega)} = |u|_{dom(L)}$.

Finally, to show that $dom(L)$ is a Hilbert space it is enough to show that it is complete. Let (u_n) be a Cauchy sequence in $dom(L)$. Then (u_n) and (Lu_n) are Cauchy sequences in $L^2(\Omega)$, so they converge in $L^2(\Omega)$ to u and v, respectively. Necessarily $v = Lu$ because for $\varphi \in \mathcal{D}(\Omega)$, we have

$$\langle v, \varphi\rangle = \lim_{n\to\infty} \langle Lu_n, \varphi\rangle = \lim_{n\to\infty} \langle u_n, L\varphi\rangle = \langle u, L\varphi\rangle = \langle Lu, \varphi\rangle,$$

which completes the proof. □

Similar to the proof of (8.2.7), one can show that

$$H^{-1}(\Omega) = \left\{ f = \sum_{i=1}^{\infty} \langle f, \varphi_n\rangle \varphi_n \in H^{-1}(\Omega), \quad \|f\|^2_{H^{-1}(\Omega)} = \sum_{n=1}^{\infty} \lambda_n^{-1} \langle f, \varphi_n\rangle^2 < \infty \right\}, \tag{8.2.8}$$

where $\langle f, v\rangle$ means the duality $H^{-1}(\Omega) \times H_0^1(\Omega)$ and the convergence of the series is in $H^{-1}(\Omega)$. We leave the proof as an exercise; see Problem 8.9.

Definition 8.2.3 *Let $F = F(s)$, $s \in \mathbb{R}$, be a function. Then we define*

$$\begin{aligned} F(L) : dom(F(L)) &\mapsto L^2(\Omega), \\ F(L)(u) &= \sum_{n=1}^{\infty} F(\lambda_n)(u, \varphi_n)\varphi_n, \quad \textit{where} \qquad (8.2.9) \\ dom(F(L)) &= \left\{ u \in L^2(\Omega), \ \sum_{n=1}^{\infty} |F(\lambda_n)|^2 |(u, \varphi_n)|^2 < \infty \right\}. \end{aligned}$$

Example 8.2.4 *For $t \geq 0$ consider $F(s) = e^{-ts}$, $s \in \mathbb{R}$, and define*

$$e^{-tL}(u) = \sum_{n=1}^{\infty} e^{-t\lambda_n}(u, \varphi_n)\varphi_n, \quad \textit{with} \tag{8.2.10}$$

$$dom(e^{-tL}) = \left\{ u \in L^2(\Omega), \quad \sum_{n=1}^{\infty} e^{-2t\lambda_n} |(u, \varphi_n)|^2 < \infty \right\}. \tag{8.2.11}$$

We close this section by introducing the following concepts and associated results. For the proof of these results and more details, the reader can see, for example, [15, 36]. Let $(Y, \|\cdot\|)$ be a Banach space, $a, b \in \mathbb{R}$, $a < b$, $k \in \mathbb{N}_0$, $p \in [1, \infty]$. We have introduced the spaces $C^k((a,b); Y)$; see Definition 1.1.6 with $\Omega = (a, b)$ and $X = \mathbb{R}$. The spaces $C^k([a,b]; Y)$ and $L^p(a, b; Y)$ are defined by

$$C^k([a,b]; Y) = \Big\{ u : [a,b] \mapsto Y, u^{(i)} \textit{ extends as a } C^0([a,b]; Y) \textit{ for all } i = 0, \ldots, k,$$

$$\textit{equipped with } \ \|u\|_{C^k([a,b];Y)} = \sum_{i=0}^{k} \sup_{t\in[a,b]} \|u^{(i)}(t)\|_Y \Big\}, \tag{8.2.12}$$

$$\begin{aligned} L^p(a,b;Y) \ &= \ \Big\{ t \in (a,b) \mapsto u(t) \in Y \textit{ measurable and } \ \|u\|_{L^p(a,b;Y)} < \infty, \ \textit{ where} \\ &\qquad \|u\|^p_{L^p(a,b;Y)} = \int_a^b \|u(t)\|^p_Y dt, \qquad \textit{if } p \in [1,\infty), \\ &\qquad \|u\|_{L^\infty(a,b;Y)} = \|\|u(t)\|_Y\|_{L^\infty(a,b)} \quad \textit{if } p = \infty \Big\}. \end{aligned} \tag{8.2.13}$$

These spaces equipped with the respective norms are Banach spaces. If Y is a separable Hilbert space with $\{\varphi_n,\ n \in \mathbb{N}\}$ an orthonormal basis, then for every $u \in L^2(a,b;Y)$ we have

$$u(t) \ = \ \sum_{n\in\mathbb{N}} (u(t),\varphi_n)_Y \varphi_n =: \sum_{n\in\mathbb{N}} u_n(t)\varphi_n, \tag{8.2.14}$$

$$\|u\|^2_{L^2(a,b;Y)} \ = \ \int_a^b \|u(t)\|^2_Y dt = \sum_{n\in\mathbb{N}} \int_a^b |u_n(t)|^2 dt, \quad u_n \in L^2(a,b). \tag{8.2.15}$$

8.3 Weak solution to the heat equation

We introduce the concept of a weak solution to the heat equation.

Definition 8.3.1 *Let $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T;H^{-1}(\Omega))$. A function u is called "weak, or variational, solution to* (8.0.1)*" if*

i) $u \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$, (8.3.1a)

ii) $u(0) = u_0$, (8.3.1b)

iii) $\dfrac{d}{dt}(u(t),v) + \langle Lu(t),v\rangle = \langle f(t),v\rangle, \quad \forall v \in H^1_0(\Omega), \quad \textit{in } \ \mathscr{D}'(0,T).$ (8.3.1c)

The problem i)–iii) is called "weak, or variational, form of (8.0.1)*".*

The decompositions (8.2.6), (8.2.7), and (8.2.8) motivate the following formal solution to (8.0.1). We assume $u_0 = \sum_{n=1}^\infty u_{0,n}\varphi_n$ and $f(t) = \sum_{n=1}^\infty f_n(t)\varphi_n$ are the decompositions of u_0 in $L^2(\Omega)$ and of f in $L^2(0,T;H^{-1}(\Omega))$, and look for $u = \sum_{n=1}^\infty u_n(t)\varphi_n$. Replacing u in (8.0.1) gives

$$\begin{aligned} \sum_{n=1}^\infty (u_n'(t) + \lambda_n u_n(t))\varphi_n(x) \ &= \ \sum_{n=1}^\infty f_n(t)\varphi_n(x), \quad t>0, \ \ x\in\Omega, \\ \sum_{n=1}^\infty u_n(0)\varphi_n \ &= \ \sum_{n=1}^\infty u_{0,n}\varphi_n, \end{aligned}$$

which imply

$$u_n'(t) \ + \ \lambda_n u_n(t) = f_n(t), \qquad u_n(0) = u_{0,n}, \quad \forall n \in \mathbb{N}. \tag{8.3.2}$$

This equation can be written equivalently as $(e^{\lambda_n t}u_n)' = e^{\lambda_n t}f_n(t)$, which has a unique solution

$$u_n \in H^1(0,T) \cap C^0([0,T]), \tag{8.3.3a}$$
$$u_n(t) = e^{-\lambda_n t}u_{0,n} + \int_0^t e^{\lambda_n(s-t)} f_n(s)ds, \tag{8.3.3b}$$

because $f_n \in L^2(0,T)$. We have this result.

Theorem 8.3.2 *Let L be as in (8.2.3) and assume (8.2.4), (8.2.5) hold, $u_0 \in L^2(\Omega)$, and $f \in L^2(0,T;H^{-1}(\Omega))$. Then the problem (8.0.1) has a unique weak solution $u \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ given by*

$$\begin{aligned} u(t)(x) &= \sum_{n=1}^{\infty}\left(e^{-\lambda_n t}u_{0,n} + \int_0^t e^{\lambda_n(s-t)} f_n(s)ds\right)\varphi_n(x) \\ &=: e^{-tL}u_0 + \int_0^t e^{(s-t)L} f(s)ds, \end{aligned} \tag{8.3.4}$$

with u_n given by (8.3.3).

Proof. Set $S_u^k(t) = \sum_{n=1}^k u_n(t)\varphi_n$, $S_0^k = \sum_{n=1}^k u_{0,n}$, and $S_f^k(t) = \sum_{n=1}^k f_n(t)\varphi_n$. It is clear that (S_0^k) converges to u_0 in $L^2(\Omega)$ and (S_f^k) converges to f in $L^2(0,T;H^{-1}(\Omega))$. We note also that from (8.2.15) and (8.2.7), for all $u \in C^0([0,T];L^2(\Omega))$, resp. $u \in L^2(0,T;H_0^1(\Omega))$, we have

$$\|S_u^k\|^2_{C^0([0,T];L^2(\Omega))} = \sup_{t\in[0,T]}\sum_{n=1}^k |u_n(t)|^2, \tag{8.3.5}$$
$$\|S_u^k\|^2_{L^2(0,T;H_0^1(\Omega))} = \sum_{n=1}^k \lambda_n \int_0^T |u_n(t)|^2 dt. \tag{8.3.6}$$

Claim: u satisfies (8.3.1a) and (8.3.1b). Indeed, multiplying (8.3.2) by $2u_n$ and integrating in $(0,t) \subset (0,T)$ gives

$$\begin{aligned} \int_0^t (u_n^2(s))'ds + 2\lambda_n \int_0^t |u_n(s)|^2 ds &= 2\int_0^t f_n(s)u_n(s)ds, \qquad \textit{which implies} \\ u_n^2(t) + 2\lambda_n \int_0^t |u_n(s)|^2 ds &\le u_{0,n}^2 + \frac{1}{\lambda_n}\int_0^t |f_n(s)|^2 ds + \int_0^t \lambda_n |u_n(s)|^2 ds. \end{aligned}$$

Therefore, for all $t \in [0,T]$ we have

$$u_n^2(t) + \lambda_n \int_0^T |u_n(s)|^2 ds \le u_{0,n}^2 + \frac{1}{\lambda_n}\int_0^T |f_n(s)|^2 ds.$$

Summing this result for n from $k+1$ to $k+l$ gives

$$\begin{aligned}\|S_u^{k+l}(t)-S_u^k(t)\|^2_{L^2(\Omega)} &+ \|S_u^{k+l}-S_u^k\|^2_{L^2((0,T);H_0^1(\Omega))}\\ &\leq \|S_0^{k+l}-S_0^k\|^2_{L^2(\Omega)}+\|S_f^{k+l}-S_f^k\|^2_{L^2((0,T);H^{-1}(\Omega))}, \qquad (8.3.7)\end{aligned}$$

which shows that (S_u^k) is a Cauchy sequence in $C^0([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, because (S_0^k) converges to u_0 in $L^2(\Omega)$ and (S_f^k) converges to f in $L^2(0,T;H^{-1}(\Omega))$. As $C^0([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ is Banach, the sequence (S_u^k) converges to u in $C^0([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$. Also from $S_u^k(0)=S_0^k$ and (S_u^k) converging to u in $C^0([0,T];L^2(\Omega))$, we get $u(0)=u_0$, which proves the claim.
Claim: u satisfies (8.3.1c). Indeed, the equation (8.3.2) is equivalent to

$$\frac{d}{dt}(S_u^k,\varphi_n)+\langle LS_u^k,\varphi_n\rangle=\langle S_f^k,\varphi_n\rangle, \quad \forall n\in\{1,2,\dots,k\}.$$

Multiplying this equation by $\psi=\psi(t)\in\mathcal{D}(0,T)$ and integrating by parts in $(0,T)$ gives

$$-\int_0^T (S_u^k,\varphi_n)\psi' dt+\int_0^T\langle LS_u^k,\varphi_n\rangle\psi dt=\int_0^T\langle S_f^k,\varphi_n\rangle\psi dt. \qquad (8.3.8)$$

Since $\lim_{k\to\infty}\|S_u^k-u\|_{L^2(0,T;H_0^1(\Omega))}=0$ and $\lim_{k\to\infty}\|S_f^k-f\|_{L^2(0,T;H^{-1}(\Omega))}=0$, we get

$$\begin{aligned}\lim_{k\to\infty}\int_0^T (S_u^k,\varphi_n)\psi' dt &= \int_0^T (u,\varphi_n)\psi' dt, \qquad (8.3.9)\\ \lim_{k\to\infty}\int_0^T \langle LS_u^k,\varphi_n\rangle\psi dt, &= \lim_{k\to\infty}\int_0^T ((S_u^k,L\varphi_n))\psi dt=\int_0^T ((u,L\varphi_n))\psi dt\\ &= \int_0^T\langle Lu,\varphi_n\rangle\psi dt, \qquad \textit{and} \qquad (8.3.10)\\ \lim_{k\to\infty}\int_0^T\langle S_f^k,\varphi_n\rangle\psi dt &= \int_0^T\langle f,\varphi_n\rangle\psi dt. \qquad (8.3.11)\end{aligned}$$

Therefore, letting $k\to\infty$ in (8.3.8) combined with (8.3.9), (8.3.11) gives

$$-\int_0^T (u,\varphi_n)\psi' dt+\int_0^T\langle Lu,\varphi_n\rangle\psi dt=\int_0^T\langle f,\varphi_n\rangle\psi dt, \quad \forall n\in\mathbb{N}. \qquad (8.3.12)$$

This equality holds with φ_n replaced by any finite combination of $\varphi_1,\varphi_2,\dots$, which by the density of $\{\varphi_n\}$ in $H_0^1(\Omega)$ implies (8.3.1c).
Claim: The solution is unique. It is enough to show that if u solves (8.3.1) with $f=0$ and $u_0=0$, then $u=0$. Indeed, we have $u=\sum_{n=1}^\infty u_n\varphi_n$ because $u\in L^2((0,T),H^1(\Omega))$. Taking $\varphi=\varphi_n$ in (8.3.1c) gives $u_n'+\lambda_n u_n=0$ and $u_n(0)=0$, which implies $u_n=0$ and so $u=0$.

8.4 Weak solution to the wave equation

We will solve the problem (8.0.2) with L given by (8.2.3) by using the same approach we used for the heat equation. Let us first introduce the concept of weak solutions to the wave equation.

Definition 8.4.1 *Given $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, and $f \in L^2(0,T;L^2(\Omega))$, a function u is called a "weak, or variational, solution to (8.0.2)", if*

$$i)\ u \in C^1([0,T];L^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)), \tag{8.4.1a}$$

$$ii)\ u(0) = u_0, \quad u'(0) = u_1, \tag{8.4.1b}$$

$$iii)\ \frac{d}{dt}(u'(t),v) + \langle Lu(t),v\rangle = (f(t),v), \quad \forall v \in H_0^1(\Omega), \quad in \quad \mathscr{D}'(0,T). \tag{8.4.1c}$$

The problem (8.4.1) is called "weak, or variational, form of (8.0.2)".

Assume the decomposition of u_0 in $H_0^1(\Omega)$, of u_1 in $L^2(\Omega)$, and of f in $L^2(0,T;L^2(\Omega))$ are given by $u_0 = \sum_{n=1}^\infty u_{0,n}\varphi_n$, $u_1 = \sum_{n=1}^\infty u_{1,n}\varphi_n$, $f(t) = \sum_{n=1}^\infty f_n(t)\varphi_n$. If $u = \sum_{n=1}^\infty u_n(t)\varphi_n$, then replacing it in (8.4.1c) yields

$$\sum_{n=1}^\infty (u_n''(t) + \lambda_n u_n(t) - f_n(t))\varphi_n(x) = 0, \quad t > 0, \ \ x \in \Omega,$$

which implies

$$u_n''(t) + \lambda_n u_n(t) = f_n(t), \quad u_n(0) = u_{0,n}, \ \ u_n'(0) = u_{1,n}. \tag{8.4.2}$$

This equation has a unique solution satisfying

$$u_n \in H^2(0,T) \cap C^1([0,T]), \tag{8.4.3a}$$

$$\begin{aligned} u_n(t) = {} & \cos\left(t\sqrt{\lambda_n}\right)u_{0,n} + \frac{1}{\sqrt{\lambda_n}}\sin\left(t\sqrt{\lambda_n}\right)u_{1,n} \\ & + \int_0^t \frac{1}{\sqrt{\lambda_n}}\sin\left((t-s)\sqrt{\lambda_n}\right)f_n(s)ds. \end{aligned} \tag{8.4.3b}$$

The formula (8.4.3b) follows from the method of variations of constants, and it implies the regularity (8.4.3a) after using the fact: for all $g \in L^1(0,T)$ and $\varphi \in \mathscr{D}(0,T)$ we have $\langle \frac{d}{dt}\int_0^t g(s)ds, \varphi\rangle = \langle g, \varphi\rangle$.

Theorem 8.4.2 *Let L be as in (8.2.3) with (8.2.4), (8.2.5) satisfied, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, and $f \in L^2(0,T;L^2(\Omega))$. Then the problem (8.4.1) has a unique solution given by*

$$u(t) = \sum_{n=1}^\infty \left(\cos(t\sqrt{\lambda_n})u_{0,n} + \frac{1}{\sqrt{\lambda_n}}\sin(t\sqrt{\lambda_n})u_{1,n}\right)\varphi_n$$

$$+ \quad \sum_{n=1}^{\infty}\left(\int_0^t \frac{1}{\sqrt{\lambda_n}}\sin((t-s)\sqrt{\lambda_n})f_n(s)ds\right)\varphi_n \tag{8.4.4}$$

$$=: \quad \cos(t\sqrt{\lambda_n})u_0 + \frac{1}{\sqrt{\lambda_n}}\sin(t\sqrt{\lambda_n})u_1 + \int_0^t \frac{1}{\sqrt{\lambda_n}}\sin((t-s)\sqrt{\lambda_n})f(s)ds, \tag{8.4.5}$$

where u_n are given by (8.4.3b).

Proof. Like in Theorem 8.3.2 the proof goes in several steps. Set $S_u^k(t) = \sum_{n=1}^k u_n(t)\varphi_n$, $S_0^k = \sum_{n=1}^k u_{0,n}$, $S_1^k = \sum_{n=1}^k u_{1,n}$, and $S_f^k(t) = \sum_{n=1}^k f_n(t)\varphi_n$. It is clear that (S_0^k) converges to u_0 in $H_0^1(\Omega)$, (S_1^k) converges to u_1 in $L^2(\Omega)$, and (S_f^k) converges to f in $L^2(0,T;L^2(\Omega))$. We note also that from (8.2.6) and (8.2.7), we have

$$\|S_u^k\|^2_{C^1([0,T];L^2(\Omega))} = \sup_{t\in[0,T]}\sum_{n=1}^k |u_n(t)|^2 + \sup_{t\in[0,T]}\sum_{n=1}^k |u_n'(t)|^2, \tag{8.4.6}$$

$$\|S_u^k\|^2_{C^0([0,T];H_0^1(\Omega))} = \sup_{t\in[0,T]}\sum_{n=1}^k \lambda_n|u_n(t)|^2. \tag{8.4.7}$$

Claim: u satisfies (8.4.1a) and (8.4.1b). Indeed, multiplying (8.3.2) by $2u_n'$ and integrating in $(0,t)\subset(0,T)$ gives

$$\int_0^t (|u_n'(s)|^2)'ds + \lambda_n\int_0^t (|u_n(s)|^2)'ds = 2\int_0^t f_n(s)u_n'(s)ds.$$

It implies

$$|u_n'(t)|^2 + \lambda_n|u_n(t)|^2 \le |u_{1,n}|^2 + \lambda_n|u_{0,n}|^2 + \int_0^t |u_n'(s)|^2ds + \int_0^t |f_n(s)|^2ds. \tag{8.4.8}$$

To proceed, we estimate u_n' in $L^2(0,T)$. From (8.4.3b) we get

$$u_n'(t) = -\sqrt{\lambda_n}\sin(t\sqrt{\lambda_n})u_{0,n} + \cos(t\sqrt{\lambda_n})u_{1,n} + \int_0^t \cos((t-s)\sqrt{\lambda_n})f_n(s)ds.$$

This implies

$$\int_0^T |u_n'(t)|^2dt \le C\left(\lambda_n u_{0,n}^2 + u_{1,n}^2 + \int_0^T |f_n(t)|^2dt\right),$$

which combined with (8.4.8) gives

$$|u_n'(t)|^2 + \lambda_n|u_n(t)|^2 \le C\left(u_{1,n}^2 + \lambda_n u_{0,n}^2 + \int_0^T |f_n(s)|^2ds\right).$$

Summing this result for n from $k+1$ to $k+l$ yields

$$\|(S_u^{k+l})'(t) - (S_u^k)'(t)\|^2_{L^2(\Omega)} + \|S_u^{k+l}(t) - S_u^k(t)\|^2_{H_0^1(\Omega))}$$

$$\begin{aligned}&\leq \|S_1^{k+l}-S_1^k\|^2_{L^2(\Omega)}+\|S_0^{k+l}-S_0^k\|^2_{H_0^1(\Omega)}\\&+ \|S_f^{k+l}-S_f^k\|^2_{L^2((0,T);L^2(\Omega))},\end{aligned} \tag{8.4.9}$$

which shows that (S_u^k) is a Cauchy in $C^1([0,T];L^2(\Omega))\cap C^0([0,T];H_0^1(\Omega))$. As this space is Banach, the sequence (S_u^k) converges to u in $C^1([0,T];L^2(\Omega))\cap C^0([0,T];H_0^1(\Omega))$. Also from $S_u^k(0)=S_0^k$ and $(S_u^k)'(0)=S_1^k$ we get $u(0)=u_0$, $u'(0)=u_1$, which proves the claim.
Claim: u satisfies (8.4.1c). The equation (8.4.2) is equivalent to

$$((S_u^k)'',\varphi_n)+\langle LS_u^k,\varphi_n\rangle=(S_f^k,\varphi_n),\quad \forall n\in\{1,2,\dots,k\}.$$

Multiplying this equation by $\psi=\psi(t)\in\mathscr{D}(0,T)$ and integrating by parts in $(0,T)$ gives

$$-\int_0^T((S_u^k)',\varphi_n)\psi' dt+\int_0^T\langle LS_u^k,\varphi_n\rangle\psi dt=\int_0^T(S_f^k,\varphi_n)\psi dt. \tag{8.4.10}$$

From $\lim_{k\to\infty}\|(S_u^k)'-u'\|_{C^0([0,T];L^2(\Omega))}=0$, $\lim_{k\to\infty}\|S_u^k-u\|_{C^0([0,T];H_0^1(\Omega))}=0$, and $\lim_{k\to\infty}\|S_f^k-f\|_{L^2(0,T;L^2(\Omega))}=0$, we get

$$\lim_{k\to\infty}\int_0^T((S_u^k)',\varphi_n)\psi' dt = \int_0^T(u',\varphi_n)\psi' dt, \tag{8.4.11}$$

$$\begin{aligned}\lim_{k\to\infty}\int_0^T\langle LS_u^k,\varphi_n\rangle\psi dt &= \lim_{k\to\infty}\int_0^T\langle S_u^k,L\varphi_n\rangle\psi dt=\int_0^T\langle u,L\varphi_n\rangle\psi dt\\&=\int_0^T\langle Lu,\varphi_n\rangle\psi dt \qquad \textit{and}\end{aligned} \tag{8.4.12}$$

$$\lim_{k\to\infty}\int_0^T(S_f^k,\varphi_n)\psi dt = \int_0^T(f,\varphi_n)\psi dt. \tag{8.4.13}$$

Therefore, letting $k\to\infty$ in (8.4.10) combined with (8.4.11), (8.4.12), and (8.4.13) gives

$$-\int_0^T(u',\varphi_n)\psi' dt+\int_0^T\langle Lu,\varphi_n\rangle\psi dt=\int_0^T(f,\varphi_n)\psi dt,\quad \forall n\in\mathbb{N}, \tag{8.4.14}$$

which like in Theorem 8.3.2 proves (8.4.1c).
Claim: The solution is unique. It is enough to show that if u solves (8.4.1) with $f=0$, $u_0=0$, and $u_1=0$ then $u=0$. Indeed, as necessarily $u(t)=\sum_{n=1}^\infty u_n(t)\varphi_n$, taking $\varphi=\varphi_n$ in (8.4.1c) gives $u_n''+\lambda_n u_n=0$ and $u_n(0)=0$, $u_n'(0)=0$, which implies $u_n=0$ and so $u=0$. □

In this chapter, we considered linear heat and wave equations (8.0.1), (8.0.2), with the main part given by the linear operator L in (8.2.3). The main ingredient of the analysis was the spectral decomposition of the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. There are theories associated with (8.0.1), (8.0.2) in a more abstract context. For more details on this topic we refer the reader to, for example, [11, 27, 30, 53].

Problems

Problem 8.1 *Using the method of the separation of variables, solve the following PDEs. For each of them, consider whether the solution is classical or not, and find* $\lim_{t\to\infty} u(x,t)$.

(a)
$$\begin{array}{lll} u_t - u_{xx} = 0 & \textit{in} & (0,1)\times(0,\infty), \\ u = 0 & \textit{on} & \{0,1\}\times(0,\infty), \\ u(x,0) = x(x-1) & \textit{on} & (0,1). \end{array}$$

(b)
$$\begin{array}{lll} u_t - u_{xx} = u & \textit{in} & (0,1)\times(0,\infty), \\ u(0,t) = u_x(1,t) = 0 & \textit{for} & t>0, \\ u(x,0) = x & \textit{on} & (0,1). \end{array}$$

(c)
$$\begin{array}{lll} u_t - u_{xx} = 0 & \textit{in} & (0,\pi)\times(0,\infty), \\ u_x(0,t) = u_x(\pi,t) & \textit{for} & t>0, \\ u(x,0) = \pi - x & \textit{on} & (0,\pi). \end{array}$$

(d)
$$\begin{array}{lll} u_t - c^2u_{xx} = 0 & \textit{in} & (0,2\pi)\times(0,\infty), \\ u(0,t) = u(2\pi,t) & \textit{for} & t>0, \\ u_x(0,t) = u_x(2\pi,t) & \textit{for} & t>0, \\ u(x,0) = g(x) & \textit{on} & (0,2\pi),\ \ g\in C^2([0,2\pi]) \textit{ and periodic.} \end{array}$$

Problem 8.2 *Consider the problem*
$$\begin{array}{lll} u_t - u_{xx} = \alpha u & \textit{in} & (0,\pi)\times(0,\infty), \\ u(0,t) = u(\pi,t) = 0 & \textit{on} & \{0,\pi\}\times(0,\infty), \\ u(x,0) = x(\pi - x) & \textit{on} & (0,\pi), \end{array}$$
with α *being a real parameter. Show that the method of separation of variables gives a classical series solution and find* $\lim_{t\to\infty} u(x,t)$ *in the case* $\alpha < 1$. *Consider the solution and* $\lim_{t\to\infty} u(x,t)$ *for other values of* α.

Problem 8.3 *Using the method of separation of variables solve the following PDEs.*

(a)
$$\begin{array}{lll} u_{tt} - u_{xx} = 0 & \textit{in} & (0,\pi)\times(0,\infty), \\ u(0,t) = u(\pi,t) = 0 & \textit{for} & t>0, \\ u(x,0) = \sin^3 x,\ \ u_t(x,t) = \sin(2x) & \textit{on} & (0,\pi). \end{array}$$

(b)
$$\begin{array}{lll} u_{tt} - u_{xx} = 0 & \textit{in} & (0,1)\times(0,\infty), \\ u(0,t) = u(1,t) = 0 & \textit{for} & t>0, \\ u(x,0) = x(1-x),\ u_t(x,0) = 0 & \textit{on} & (0,1). \end{array}$$

In both cases, discuss whether the series solution is a classical solution or not.

Problem 8.4 *Let* $u_0(x)$, $u_1(x)$, $f(x,t)$ *be given smooth functions. Show that*

$$u(x,t) = \frac{1}{2}(u_0(x-ct) + u_0(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(s)ds$$

$$+ \quad \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(r,s)drds$$

is a classical solution to

$$u_{tt} - c^2 u_{xx} = f \ \text{ in } \ \mathbb{R}\times(0,\infty), \qquad u(\cdot,0) = u_0(\cdot), \ u_t(\cdot,0) = u_1(\cdot) \ \text{ on } \ \mathbb{R}.$$

Problem 8.5 *Find d'Alembert formula in Problem 8.4 by using the formula (3.0.3) for* $v_t - v_x = f$ *and* $u_t + u_x = v$.

Problem 8.6 *Consider the wave equation*

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f \ \text{ in } \ (0,\infty)\times(0,\infty), \\ u = u_0, \ u_t &= u_1 \text{ on } \ (0,\infty)\times\{0\}, \\ u &= 0 \ \text{ on } \ \{0\}\times(0,\infty), \end{aligned}$$

with $u_0(x)$, $u_1(x)$, $f(x,t)$ *given smooth functions,* $u_0(0) = u_1(0) = 0$ *and* $u_0'(0^+)$, $u_1'(0^+)$ *exist. Extend* u_0, u_1, *and* $f(\cdot,t)$ *in* $\mathbb{R}$ *as odd functions and then use the formula of Problem 8.4 above to find a classical solution* $u(x,t)$.

Problem 8.7 *Consider the wave equation*

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f \ \text{ in } \ (0,\ell)\times(0,\infty), \\ u = u_0, \ u_t &= u_1 \text{ on } \ (0,\ell)\times\{0\}, \\ u &= 0 \ \text{ on } \ \{0,\ell\}\times(0,\infty), \end{aligned}$$

with $u_0(x)$, $u_1(x)$, $f(x,t)$ *given smooth functions with* $u_0(0) = u_1(0) = 0$ *and* $u_0'(0^+)$, $u_1'(0^+)$ *well-defined. Extend* u_0, u_1, *and* $f(\cdot,t)$ *in* $(-\ell,\ell)$ *as odd functions and next in* $\mathbb{R}$ *as* 2ℓ *periodic functions, and then use the formula of Problem 8.4 above to find a classical solution* u.

Problem 8.8 *Use d'Alembert formula to solve* $u(x,t)$ *and* t_0.

(a) $u_{tt} - 4u_{xx} = 0$ *in* $\mathbb{R}\times(0,\infty)$, *with* $u(\cdot,0) = \mathbb{1}_{(-1,1)}$, $u_t(\cdot,0) = \mathbb{1}_{(0,1)}$, *and* t_0 *solution to* $u(2,t_0) = \max\{u(2,t),\ t\geq 0\}$.

(b) $u_{tt} - 9u_{xx} = 0$ *in* $\mathbb{R}\times(0,\infty)$, *with* $u(\cdot,0) = 5\cdot\mathbb{1}_{[-1,1]}$, $u_t(\cdot,0) = \mathbb{1}_{[-1,1]}$. *Here, for a given* x_0, t_0 *solves* $u(x_0,t_0) = \max\{u(x_0,t),\ t\geq 0\}$.[1]

Problem 8.9 *Prove (8.2.8). As an application, take* $Lu = -u''$ *in* $\Omega = (0,2\pi)$ *and write* $\delta_\pi(x) := \delta_0(x-\pi)$ *as a series in terms of eigenfunctions of* L.

[1] This problem has an application: if u is the pressure after a shock wave generated by an explosion and a building located at the position x_0 supports a pressure up to 4 (u is a dimensionless variable), then this building will collapse or not depend on the sign of $\delta := 4 - \max\{u(x_0,t),\ t>0\}$—it collapses if $\delta < 0$ and it does not if $\delta \geq 0$.

Problem 8.10 *Consider the problem*

$$\begin{aligned} u' - u_{xx} &= f \quad &&\textit{in} \quad \Omega \times (0,T),\ \Omega = (0, 2\pi), \\ u(\cdot, t) &= 0 \quad &&\textit{on} \quad \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad &&\textit{on} \quad \Omega. \end{aligned}$$

with $u_0 \in L^2(\Omega)$, $f(x,t) = \delta_\pi(x)$. *Prove that the series solution* (8.3.4) *associated with this problem converges in* $C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$ *and satisfies* (8.3.1c). *Prove moreover that* $u' \in L^2(0,T; H^{-1}(\Omega))$.

Problem 8.11 *Let* u *be the solution to the problem* (8.0.1) *as given by Theorem 8.3.2.*

(a) Prove that $u \in H^1(0,T; H^{-1}(\Omega))$ *and therefore* (8.0.1a) *holds in* $L^2(0,T; H^{-1}(\Omega))$.

(b) Prove that if $u_0 \in H^1_0(\Omega)$ *then* $u \in C^0([0,T]; H^1_0(\Omega))$.

Problem 8.12 *Let* u *be the solution to the problem* (8.0.1) *as given by Theorem 8.3.2. By using the density of* $\{\zeta(x)\psi(t),\ \zeta \in \mathcal{D}(\Omega),\ \ \psi \in \mathcal{D}(0,T)\}$ *in* $\mathcal{D}(\Omega \times (0,T))$ *show that* u *satisfies*

$$u' + Lu = f \ \ \textit{in} \ \ \mathcal{D}'(\Omega \times (0,T)).$$

Problem 8.13 *Show that the problem* (8.0.1) *with* (8.0.1b) *replaced by* $u(0) = u(T)$ *has a unique solution in* $C^0_T([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$ *in the sense of Definition 8.3.1 (with* $u(0) = u_0$ *replaced by* $u(0) = u(T)$*) for every* $f \in C^0_T([0,T]; L^2(\Omega)) := \{g \in C^0([0,T]; L^2(\Omega)),\ g(0) = g(T)\}$.

Problem 8.14 *Let* u *be the solution to* (8.0.2) *as given by Theorem 8.4.2. Prove that*

$$\begin{aligned} E(t) &:= \int_\Omega |u'(x,t)|^2 dx + \langle Lu(\cdot,t), u(\cdot,t)\rangle \\ &= \int_\Omega |u_1(x)|^2 dx + \langle Lu_0, u_0\rangle + 2\int_0^t \int_\Omega f(x,s)u'(x,s)dxds, \quad t \in (0,T). \end{aligned} \tag{8.4.15}$$

Hence, if $f = 0$ *in a certain interval* $(a,b) \subset (0,T)$ *then* $E(t)$ *remains constant in* (a,b).

9. Annex

In this chapter, we will collect different results, with or without proof, which complement the material considered in the previous chapters.

9.1 Notations and review (Ch. 1)

9.1.1 Continuous differentiable functions

Proposition 9.1.1 *Let $\Omega \subset \mathbb{R}^N$ be open, $k \in \mathbb{N}_0$, $\lambda \in (0,1]$. Define $C_b^k(\Omega)$ and $C_b^{k,\lambda}(\Omega)$ spaces as in Proposition 1.1.11, and endow them with the norms $\|\cdot\|_{C_b^k(\Omega)}$, $\|\cdot\|_{C_b^k(\Omega)}$, respectively. These spaces are Banach spaces.*

Proof. It is easy to show that $\|u\|_{C_b^k(\Omega)}$ and $\|u\|_{C_b^{k,\lambda}(\Omega)}$ are norms, which we leave as an exercise. Now we show that the spaces are Banach.
i) *Claim: $C_b^0(\Omega)$ is a Banach space.* Let (u_n) be a Cauchy sequence in $C_b^0(\Omega)$, so for given $\epsilon > 0$ there exists n_ϵ such that

$$|u_{n+m}(x) - u_n(x)| < \epsilon, \qquad \forall n, m \in \mathbb{N}, \quad n > n_\epsilon, \quad \forall x \in \Omega. \tag{9.1.1}$$

We have to show that (u_n) converges to a certain $u \in C_b^0(\Omega)$ in $C_b^0(\Omega)$ norm. From (9.1.1), it follows that $(u_n(x))$ is uniformly bounded, Cauchy in $\mathbb{R}$ and converges to a certain $u(x)$, with necessarily $\{u(x),\ x \in \Omega\}$ bounded. For $n > n_\epsilon$ and $x \in \Omega$, we obtain

$$|u_n(x) - u(x)| = \lim_{m\to\infty} |u_n(x) - u_{n+m}(x)| \le \epsilon, \tag{9.1.2}$$

which shows that (u_n) converges uniformly[1] to u in Ω. To complete the proof of the claim, it is enough to show $u \in C^0(\Omega)$. For $x, x+h \in \Omega$, we have

$$|u(x+h) - u(x)| \quad \le \quad |u(x+h) - u_n(x+h)| + |u(x) - u_n(x)| + |u_n(x+h) - u_n(x)|,$$

which combined with the uniform convergence of (u_n), of the uniform continuity of u_n near[2] x and of (9.1.1) implies $u \in C^0(\Omega)$ and proves the claim.

[1]A sequence (u_n) in $C^0(\Omega; Y)$, $\Omega \subset X$, "*converges uniformly to $u : \Omega \mapsto Y$ in Ω*" if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for all $n > n_\epsilon$ and all $x \in \Omega$ we have $\|u_n(x) - u(x)\|_Y < \epsilon$.

[2]It is easy to prove that if $v \in C^0(\Omega)$ then v is uniformly continuous in $\overline{B}(x, r_x) = \{y \in \Omega, |y - x| \le r_x\} \subset \Omega$.

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6_9

ii) Claim: $C_b^k(\Omega)$ is a Banach space. Proceeding as in i) above, we show that given a Cauchy sequence (u_n) in $C_b^1(\Omega)$, there exist $u, v^i \in C_b^0(\Omega), i=1,\ldots,N$, with $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} \partial_i u_n = v^i$ in $C_b^0(\Omega)$. Necessarily, $\partial_i u = v^i$ because for $x \in \Omega$ and $|t|$ small we have

$$\begin{aligned}
\frac{1}{t}(u(x+te_i)-u(x)) &= \frac{1}{t}\lim_{n\to\infty}(u_n(x+te_i)-u_n(x)| \\
&= \lim_{n\to\infty}\int_0^1 \partial_i u_n(x+ste_i)ds = \int_0^1 v^i(x+ste_i)ds \\
&\xrightarrow[t\to 0]{} v^i(x),
\end{aligned}$$

which completes the claim for $C_b^1(\Omega)$ spaces. The proof for $C_b^k(\Omega)$ spaces follows similarly by recurrence.
iii) Claim: $C_b^{k,\lambda}(\Omega)$ is a Banach space. Given a Cauchy sequence (u_n) in $C_b^{k,\lambda}(\Omega)$, from cases above we get the convergence of (u_n) to a certain u in $C_b^k(\Omega)$. Furthermore, as $(\|u_n\|_{C_b^{k,\lambda}(\Omega)})$ is necessarily bounded, for $x, y \in \Omega$ with $x \neq y$, $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = k$, we have

$$\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda} = \lim_{n\to\infty}\frac{|D^\alpha u_n(x) - D^\alpha u_n(y)|}{|x-y|^\lambda} \leq \lim_{n\to\infty}\|u_n\|_{C_b^{k,\lambda}(\Omega)} < \infty,$$

and therefore $|u|_{C^{k,\lambda}(\Omega)} < \infty$, so $u \in C^{k,\lambda}(\Omega)$. Finally for $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{|D^\alpha(u-u_n)(x) - D^\alpha(u-u_n)(y)|}{|x-y|^\lambda} &= \lim_{m\to\infty}\frac{|D^\alpha(u_{n+m}-u_n)(x) - D^\alpha(u_{n+m}-u_n)(y)|}{|x-y|^\lambda} \\
&\leq \lim_{m\to\infty}\|u_{n+m}-u_n\|_{C^{k,\lambda}(\Omega)},
\end{aligned}$$

which from the Cauchy property of (u_n) in $C_b^{k,\lambda}(\Omega)$ implies $\lim_{n\to\infty} u_n = u$ in $C_b^{k,\lambda}(\Omega)$, and completes the proof.

9.1.2 Some results from $L^p(\Omega)$ spaces

Theorem 9.1.2 (Tonelli) *Let $N,\ M \in \mathbb{N}$ and $A \subset \mathbb{R}^N$, $B \subset \mathbb{R}^M$ be two open sets, and $u : A \times B \mapsto \mathbb{R}$ be a measurable function such that*

$$\int_B |u(x,y)|dy < \infty \quad \textit{for a.a. } x \in A, \quad \textit{and} \quad \int_A dx \int_B |u(x,y)|dy < \infty.$$

Then $u \in L^1(A \times B)$.

The following is a sort of the reciprocal from the theorem above.

Theorem 9.1.3 (Fubini) *Let $u \in L^1(A \times B)$ with A, B as in Theorem 9.1.2. Then*

$$i) \quad \text{for a.a. } x \in A, \quad u(x,\cdot) \in L^1(B), \text{ and } x \to \int_B u(x,y)dy \in L^1(A),$$

$$ii) \quad \text{for a.a. } y \in B, \quad u(\cdot,y) \in L^1(A), \text{ and } y \to \int_A u(x,y)dx \in L^1(B),$$

$$iii) \quad \int_A \int_B u(x,y)dydx = \int_B \int_A u(x,y)dxdy = \iint_{A\times B} u(x,y)dxdy.$$

9.1.3 An application: Ordinary Differential Equations

Theorem 9.1.4 *Let $(X, \|\cdot\|_X)$ be a Banach space, $U \subset X$ open and $f : U \mapsto X$ a Lipschitz function with Lipschitz constant L. Let $x_0 \in U$, $\rho \in (0,1)$ such that $\overline{B}(x_0, 2\rho) \subset U$ and $K > 0$ such that $\|f\|_{C^0(\overline{B}(x_0,2\rho))} \leq K$. Finally, let $0 < r < \min\{1/L, \rho/K\}$ and set $\overline{I}_r = [-r, r]$. Then there exists $\alpha : \overline{I}_r \times B(x_0,\rho) \mapsto U$, such that $t \in \overline{I}_r \mapsto \alpha(t,x) \in X$ is the unique solution to (1.3.4) in $C^1(\overline{I}_r; X)$.*

Proof. The proof of this theorem is an application of Theorem 1.3.10. It is simple to show that (1.3.4) in $\overline{I}_r$ is equivalent to

$$\text{find } \alpha(\cdot,x) \in C^1(\overline{I}_r; X), \quad \alpha(t,x) = x + \int_0^t f(\alpha(s,x))ds, \quad t \in \overline{I}_r. \tag{9.1.3}$$

Now consider the set

$$M = \{z \in C^0(\overline{I}_r; \overline{B}(x_0, 2\rho)), \qquad z(0) \in \overline{B}(x_0,\rho)\}.$$

Equipped with the distance $d(y,z) := \|y(\cdot) - z(\cdot)\|_{C^0(\overline{I}_r;X)}$, M is a complete metric space. Next, for $x \in \overline{B}(x_0,\rho)$ consider the map

$$T_x : M \mapsto M, \quad (T_x z)(t) = x + \int_0^t f(z(s))ds, \; t \in \overline{I}_r, \quad x \in \overline{B}(x_0,\rho).$$

Clearly T_x is well-defined, because $T_x z \in C^0(\overline{I}_r; X)$, $(T_x z)(0) = x \in \overline{B}(x_0,\rho)$, and for $t \in \overline{I}_r$ we have $(T_x z)(t) \in \overline{B}(x_0, 2\rho)$, because

$$\|(T_x z)(t) - x_0\|_X \leq \|x - x_0\|_X + \int_0^t \|f(z(s))\|_X ds \leq \rho + rK \leq 2\rho.$$

Then every solution to (9.1.3) is a fixed point of T_x. Note T_x is a contraction because

$$\begin{aligned} \|T_x z_1 - T_x z_2\|_{C^0(\overline{I}_r;X)} &\leq \sup\left\{\int_0^t \|f(z_1(s)) - f(z_2(s))\|_X ds, \; t \in \overline{I}_r\right\} \\ &\leq \kappa \|z_1 - z_2\|_{C^0(\overline{I}_r;X)}, \quad 0 \leq \kappa = rL < 1. \end{aligned} \tag{9.1.4}$$

Therefore, from Theorem 1.3.10 there exists a unique fixed point $\alpha_x \in Z$ of T_x, and $\alpha(t,x) = \alpha_x(t)$ is a solution to (9.1.3).

As f is Lipschitz from U in X, it follows that $f \in C^0(U;X)$ and therefore from (9.1.3) we obtain $\alpha(\cdot,x) \in C^1(\overline{I}_r;U)$ and $\alpha'(t,x) = f(\alpha(t,x))$, $t \in \overline{I}_r$. Finally, if $f \in C^k(U;X)$, from $\alpha'(t,x) = f(\alpha(t,x))$ we get $\alpha(\cdot,x) \in C^{k+1}(\overline{I}_r;X)$. □

The uniqueness of the solution to (1.3.4) holds as follows.

Theorem 9.1.5 *Assume the conditions of Theorem 9.1.4 hold. Let $x \in U$ and assume $\alpha_1(\cdot,x) \in C^1(J_1;U)$ and $\alpha_2(\cdot,x) \in C^1(J_2;U)$ solve (1.3.4), where J_1, I_2 are closed intervals containing 0. Then $\alpha_1(\cdot,x) = \alpha_2(\cdot,x)$ on $J_1 \cap J_2$.*

Proof. See [28, Chapter XIV, §3]. □

When f is Lipschitz, we can prove by direct calculations that the solution α is Lipschitz with respect to x.

Theorem 9.1.6 *Let $x \in \overline{B}(x_0,\rho)$ and $\alpha(\cdot,x) \in C^1(\overline{I}_r,U)$ as given by Theorem 9.1.4. Then $x \mapsto \alpha(\cdot,x)$ is uniformly Lipschitz from $\overline{B}(x_0,\rho)$ to $C^0(\overline{I}_r;X)$.*

Proof. Let $x,y \in \overline{B}(x_0,\rho)$ and $\alpha(\cdot,x)$, $\alpha(\cdot,y)$ be the corresponding $C^1(\overline{I}_r;X)$ solutions of (1.3.4a) as given by Theorem 9.1.4. We have to prove

$$\|\alpha(\cdot,x) - \alpha(\cdot,y)\|_{C^0(\overline{I}_r;X)} \leq C\|x-y\|_X,$$

for certain $C > 0$ independent of x and y. Note that from Theorem 1.3.10, $\lim_{n\to\infty} \|T_x^n\alpha(\cdot,y) - \alpha(\cdot,x)\|_{C^0(\overline{I}_r;X)} = 0$, where $T_x : Z \mapsto Z$ is defined in Theorem 9.1.4. As $T_y\alpha(\cdot,y) = \alpha(\cdot,y)$ and $\|T_xz_1 - T_xz_2\|_{C^0(\overline{I}_r;X)} \leq \kappa\|z_1 - z_2\|_X$, $0 \leq \kappa = rL < 1$ (see (9.1.4)), we have

$$\begin{array}{rcl}
\|T_x\alpha(\cdot,y) - \alpha(\cdot,y)\|_{C^0(\overline{I}_r;X)} & = & \|T_x\alpha(\cdot,y) - T_y\alpha(\cdot,y)\|_{C^0(\overline{I}_r;X)} = \|x-y\|_X, \\
\|T_x^k\alpha(\cdot,y) - T_x^{k-1}\alpha(\cdot,y)\|_{C^0(\overline{I}_r;X)} & \leq & \kappa\|T_x^{k-1}\alpha(\cdot,y) - T_x^{k-2}\alpha(\cdot,y)\|_{C^0(\overline{I}_r;X)}\|x-y\|_X \\
& \cdots & \\
& \leq & \kappa^{k-1}\|T_x\alpha(\cdot,y) - \alpha(\cdot,y)\|_{C^0(\overline{I}_r;X)}\|x-y\|_X \\
& = & \kappa^{k-1}\|x-y\|_X; \quad \textit{hence} \\
\|T_x^n\alpha(\cdot,y) - \alpha(\cdot,y)\|_{C^0(\overline{I}_r;X)} & \leq & \displaystyle\sum_{k=1,\dots,n} \|T_x^k\alpha(\cdot,y) - T_x^{k-1}\alpha(\cdot,y))\|_{C^0(\overline{I}_r;X)} \\
& \leq & (1 + \kappa + \cdot + \kappa^{n-1})\|x-y\|_X \\
& \leq & \dfrac{1}{1-\kappa}\|x-y\|_X,
\end{array}$$

which after letting n to infinity gives $\|\alpha(\cdot,x) - \alpha(\cdot,y)\|_{C^0(\overline{I}_r;X)} \leq C\|x-y\|_X$ with $C = 1/(1-\kappa)$.

9.2 First-order PDEs: classical and weak solutions (Ch. 3)

9.2.1 Classical local solutions to first-order PDEs

Theorem 9.2.1 *Assume that E and g are C^k functions, $k \geq 2$, and $(x^0, z_0(x^0), p^0(x^0))$ is non-characteristic, $x^0 \in \Gamma$. Then there exist $\rho_0 > 0$, $r_0 > 0$ such that*

(i) There exists $y^0 \mapsto p^0(y^0) \in C^{k-1}(\Gamma_0; \mathbb{R}^N)$ such that $(y^0, z_0(y^0), p^0(y^0))$ is non-characteristic for all $y^0 \in \Gamma_0 := \Gamma \cap \overline{B}(x^0, \rho_0)$.

(ii) For every $y^0 \in \Gamma_0$, the initial value problem (3.2.10) has a unique $C^k([-r_0, r_0]; \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ solution. Furthermore, if T is defined by

$$\begin{array}{llll} T: & [0, r_0] \times \Gamma_0 =: U_0 & \mapsto & \Omega_0 := T(U_0) \subset \overline{\Omega}, \\ & (t, y^0) & \mapsto & T(t, y^0) := y(t, y^0), \end{array}$$

then T is C^{k-1}, invertible, and its inverse is C^{k-1}.

(iii) The function $u : \Omega_0 \mapsto \mathbb{R}$, $u(x) = z \circ T^{-1}(x) = z(y(t, y^0))$, is a C^k solution to (3.2.11).

(iv) If $\partial_{p_N} E(x^0, z_0(x^0), (p^0_1(x^0), \ldots, p^0_{N-1}(x^0), q))$ does not change sign (remains positive or negative) for all $q \in \mathbb{R}$, then (3.2.11) has a unique C^k solution.

Proof. The proof of this theorem is not difficult, but long and technical, and will be achieved in several steps. The proof we present follows [6, 9, 16].
Proof of (i). The existence of the map p^0 follows from Lemma 3.2.2.
Proof of (ii). As E is C^k, Theorem 1.3.13 implies that the problem (3.2.10) has a unique C^k solution $(y(t, y^0), z(t, y^0), p(t, y^0))$, $t \in [-r_0, r_0]$, for arbitrary $y^0 \in \Gamma_0$. Note that, a priori, r_0 depends on y^0. However, if $\rho_0 > 0$ is small enough the constants K and L of Theorem 1.3.13 can be chosen independent of $y^0 \in \Gamma_0$, and then the existence of $r_0 > 0$ independent of y^0 follows.

Furthermore, as E and g are C^k, Theorem 1.3.15 implies $z \in C^{k-1}([-r_0, r_0] \times \Gamma_0)$ and $y, p \in C^{k-1}([-r_0, r_0] \times \Gamma_0; \mathbb{R}^N)$, provided r_0 and ρ_0 are small enough.

Next, we consider

$$\begin{array}{llll} T: & \mathbb{R} \times \mathbb{R}^{N-1} \times \{0\} & \mapsto & \mathbb{R}^N, \\ & (t, y^0) & \mapsto & T(t, y^0) := y(t, y^0) = y^0 + \int_0^t \nabla_p E(y(s, y^0), z(s, y^0), p(s, y^0)) ds. \end{array}$$

The map T is well-defined and C^{k-1} near $(0, x^0)$ in $\mathbb{R} \times \mathbb{R}^{N-1} \times \{0\}$. Furthermore, from $y(0, x^0) = x^0$ and $y' = \nabla_p E(y, z, p)$ we get

$$\left[\frac{\partial T(0, x^0)}{\partial(t, y^0)} \right] = \begin{bmatrix} \partial_{p_1} E(x^0, z_0, p^0) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{p_{N-1}} E(x^0, z_0(x^0), p^0(x^0)) & 0 & \cdots & 1 \\ \partial_{p_N} E(x^0, z_0(x^0), p^0(x^0)) & 0 & \cdots & 0 \end{bmatrix}.$$

Clearly, $det\left[\dfrac{\partial T(0,x^0)}{\partial(t,y^0)}\right] = \pm\partial_{p_N}E(x^0, z_0(x^0), p^0(x^0)) \neq 0$. Then from (inverse mapping) Theorem 1.3.11, it follows that T is C^{k-1} and invertible near $(0,x^0)$, with T^{-1} of class C^{k-1}, which completes the proof of (ii).
Proof of (iii). Clearly $u \in C^{k-1}(\Omega_0)$ and $u(y^0) = g(y^0)$ on Γ_0. We will complete the proof of (iii) in several steps.
Claim: For all $(t,y^0) \in U_0$ and $x = T(t,y^0)$, we have

$$E(x,u(x),p(t,y^0)) = 0. \tag{9.2.1}$$

Indeed, for fixed y^0 set $\epsilon(t) = E(y(t,y^0), z(t,y^0), p(t,y^0))$, $t \in [0,r_0]$. Then we have

$$\begin{aligned}\epsilon(0) &= E(y(0,y^0), z(0,y^0), p(0,y^0)) = E(y^0, z_0(y^0), p^0(y^0)) \quad (from\ (3.2.8))\\ &= 0,\\ \epsilon'(t) &= \nabla_p E(y,z,p)\cdot p' + \partial_z E(y,z,p)z' + \nabla_x E(y,z,p)\cdot y' \quad (from\ (3.2.10))\\ &= 0.\end{aligned}$$

Then Theorem 1.3.13 implies $\epsilon = 0$ and proves (9.2.1).
Claim: For all $(t,y^0) \in U_0$ and $x = T(t,y^0)$, we have

$$\nabla u(x) = p(t,y^0). \tag{9.2.2}$$

Indeed, from $x = y(t,y^0)$ and (3.2.10) we have $\dfrac{\partial z}{\partial t} = p\cdot\dfrac{\partial x}{\partial t}$, which gives

$$\frac{\partial u}{\partial x_i} = \frac{\partial z}{\partial t}\frac{\partial t}{\partial x_i} + \sum_{j=1}^{N-1}\frac{\partial z}{\partial y_j^0}\frac{\partial y_j^0}{\partial x_i} = \sum_{k=1}^{N} p_k\frac{\partial x_k}{\partial t}\frac{\partial t}{\partial x_i} + \sum_{j=1}^{N-1}\frac{\partial z}{\partial y_j^0}\frac{\partial y_j^0}{\partial x_i}. \tag{9.2.3}$$

If we admit for a moment that

$$\frac{\partial z}{\partial y_j^0} = \sum_{k=1}^{N} p_k\frac{\partial x_k}{\partial y_j^0}, \quad j = 1,\dots,N-1, \tag{9.2.4}$$

then (9.2.3) implies

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{N} p_k\left(\frac{\partial x_k}{\partial t}\frac{\partial t}{\partial x_i} + \sum_{j=1}^{N-1}\frac{\partial x_k}{\partial y_j^0}\frac{\partial y_j^0}{\partial x_i}\right) = \sum_{k=1}^{N} p_k\frac{\partial x_k}{\partial x_i} = p_i,$$

which proves (9.2.2). Now we focus on the proof of (9.2.4), which is quite technical and relies on (3.2.10). Set $\epsilon(t) = \dfrac{\partial z(t,y^0)}{\partial y_j^0} - \sum_{k=1}^{N} p_k(t,y^0)\dfrac{\partial x_k(t,y^0)}{\partial y_j^0}$. Then

$$\epsilon(0) = \frac{\partial z(0,y^0)}{\partial y_j^0} - \sum_{k=1}^{N} p_k(0,y^0)\frac{\partial x_k(0,y^0)}{\partial y_j^0} = \frac{\partial z_0}{\partial y_j^0} - p_j^0 = 0, \tag{9.2.5}$$

and using $\dfrac{\partial^2 z}{\partial t \partial y_j^0} = \displaystyle\sum_{k=1}^{N} \frac{\partial}{\partial y_j^0}\left(p_k \frac{\partial x_k}{\partial t}\right) = \sum_{k=1}^{N} \frac{\partial p_k}{\partial y_j^0}\frac{\partial x_k}{\partial t} + p_k \frac{\partial^2 x_k}{\partial t \partial y_j^0}$ gives

$$
\begin{aligned}
\epsilon'(t) &= \frac{\partial^2 z}{\partial t \partial y_j^0} - \sum_{k=1}^{N}\left(\frac{\partial p_k}{\partial t}\frac{\partial x_k}{\partial y_j^0} + p_k \frac{\partial^2 x_k}{\partial t \partial y_j^0}\right) \\
&= \sum_{k=1}^{N}\left(\frac{\partial p_k}{\partial y_j^0}\frac{\partial x_k}{\partial t} + p_k \frac{\partial^2 x_k}{\partial t \partial y_j^0}\right) - \left(\frac{\partial p_k}{\partial t}\frac{\partial x_k}{\partial y_j^0} + p_k \frac{\partial^2 x_k}{\partial t \partial y_j^0}\right) \\
&= \sum_{k=1}^{N}\left(\frac{\partial p_k}{\partial y_j^0}\frac{\partial x_k}{\partial t} - \frac{\partial p_k}{\partial t}\frac{\partial x_k}{\partial y_j^0}\right) \qquad \textit{(use (3.2.10))} \\
&= \sum_{k=1}^{N}\left(\frac{\partial p_k}{\partial y_j^0}\frac{\partial E}{\partial p_k} - \left(-\frac{\partial E}{\partial x_k} - p_k \frac{\partial E}{\partial z}\right)\frac{\partial x_k}{\partial y_j^0}\right). \qquad (9.2.6)
\end{aligned}
$$

Differentiating $E(y(t,y^0), z(t,y^0), p(t,y^0)) = 0$ (which is (9.2.1)) with respect to y_j^0 gives

$$
\sum_{k=1}^{N} \frac{\partial E}{\partial p_k}\frac{\partial p_k}{\partial y_j^0} = -\left(\sum_{k=1}^{N} \frac{\partial E}{\partial x_k}\frac{\partial x_k}{\partial y_j^0} + \frac{\partial E}{\partial z}\frac{\partial z}{\partial y_j^0}\right).
$$

Replacing this equality in (9.2.6) yields

$$
\epsilon'(t) = -\frac{\partial E}{\partial z}\left(\frac{\partial z}{\partial y_j^0} - \sum_{k=1}^{N} p_k \frac{\partial x_k}{\partial y_j^0}\right) = -\frac{\partial E}{\partial z}\cdot \epsilon(t). \qquad (9.2.7)
$$

From (9.2.5), (9.2.7), and Theorem 1.3.13, we get $\epsilon = 0$, which proves (9.2.4) and completes the proof of (9.2.2).

Claim: $u \in C^k(\Omega_0)$ and solve (3.2.11). Indeed, from (9.2.2) we have $\nabla u(x) = p(T^{-1}(x))$, which implies $\nabla u \in C^{k-1}(\overline{\Omega}_0; \mathbb{R}^N)$, so u in $C^k(\overline{\Omega}_0)$. The equations (9.2.1), (9.2.2), and $u(y^0) = z(0, y^0) = g(y^0)$, $y^0 \in \Gamma_0$ prove that u solves (3.2.11).

Proof of (iv): Proposition 3.1.1 and (iii) of this theorem set a correspondence between local C^2 solutions u to (3.0.1) and the solutions (y, z, p) of (3.2.10). Therefore, from the uniqueness of solutions to ODEs (from point (ii) of this theorem), it is enough to prove that near x^0, the non-characteristic initial conditions are unique.

The compatibility condition (3.2.7) near x^0 implies that the initial conditions y^0, $z_0(y^0)$, and $(p_1^0(y^0), \ldots, p_{N-1}^0(y^0)) = (\partial_1 g(y^0), \ldots, \partial_{N-1} g(y^0))$ are uniquely defined. The condition $E(x^0, z_0(x^0), (p_1^0(x^0), \ldots, p_{N-1}^0(x^0), p_N^0)) = 0$ implies that if p_N^0 is not unique then necessarily $\partial_{p_N} E(x^0, z_0(x^0), (p_1^0(x^0), \ldots, p_{N-1}^0(x^0), q)) = 0$ for a certain q, which contradicts the assumption and proves that the initial condition $(x^0, z_0(x^0), p^0(x^0))$ is uniquely defined. Lemma 3.2.2 shows that $(y^0, z_0(y^0), p^0(y^0))$ are uniquely defined near x^0, which completes the proof. □

9.2.2 Conservation laws and weak solutions

Proposition 9.2.2 *Let $f \in C^1(\mathbb{R})$, $g \in C^0(\mathbb{R})$. If $u \in C^1(\mathbb{R}\times\mathbb{R}^+) \cap C^0(\mathbb{R}\times[0,\infty))$ is a classical solution to (3.3.1), then u is a weak solution to (3.3.1). Reciprocally, assume u is a weak solution to (3.3.1) and $u \in C^1(\Omega)\cap C^0(\overline{\Omega})$, where $\Omega \subset \mathbb{R}\times(0,\infty)$ is open. Then u is a classical solution to (3.3.1a) in Ω and satisfies $u = g$ on $\partial\Omega \cap \{x_2 = 0\}$.*

Proof.

Assume $u \in C^1(\mathbb{R}\times\mathbb{R}^+)\cap C^0(\mathbb{R}\times[0,\infty))$ is a classical solution to (3.3.1). So u satisfies

$$0 = \partial_2 u + \partial_1(f(u)).$$

For $\varphi \in \mathcal{D}(\mathbb{R}^2)$, let $B_R^+ = \{(x_1,x_2),\ x_1^2 + x_2^2 < R^2,\ x_2 > 0\}$ with R large enough so that $\text{supp}(\varphi)\cap\{x_2>0\} \subset B_R^+$; see Fig. 9.2.1. If ν^+ is the outward unit normal on ∂B_R^+, using Gauss theorem in B_R^+ implies

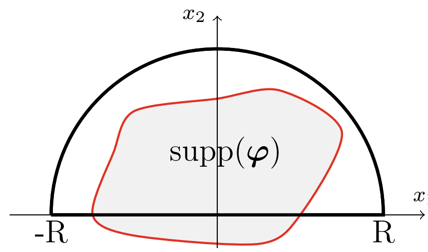

Figure 9.2.1: B_R^+ and supp(φ)

$$\begin{aligned}
0 &= \int_{B_R^+} (\partial_2 u + \partial_1(f(u)))\,\varphi dx \\
&= \int_{\partial B_R^+} \varphi(u\nu_2^+ + f(u)\nu_1^+)ds - \int_{B_R^+}(u\partial_2\varphi + f(u)\partial_1\varphi)dx \\
&= \int_{\partial B_R^+\cap\{x_2=0\}} \varphi(u\nu_2^+ + f(u)\nu_1^+)dx_1 + \int_{\partial B_R^+\cap\{x_2>0\}} \varphi(u\nu_2^+ + f(u)\nu_1^+)ds \\
&- \int_{\mathbb{R}^+}\int_{\mathbb{R}} (f(u)\partial_1\varphi + u\partial_2\varphi)\,dx_1dx_2 \\
&= -\int_{\mathbb{R}} \varphi(x_1,0)g(x_1)dx_1 - \int_{\mathbb{R}^+}\int_{\mathbb{R}} (f(u)\partial_1\varphi + u\partial_2\varphi)\,dx_1dx_2,
\end{aligned}$$

which proves the first part of the claim.

Now we assume $u \in C^1(\Omega)\cap C^0(\overline{\Omega})$ is a solution to (3.3.5). For $\varphi \in \mathcal{D}(\Omega)$, we get

$$\begin{aligned}
0 &= \int_{\mathbb{R}^+}\int_{\mathbb{R}} (u\partial_2\varphi + f(u)\partial_1\varphi)\,dx_1dx_2 + \int_{\mathbb{R}} g(x_1)\varphi(x_1,0)dx_1 \\
&= \int_{\Omega} (u\partial_2\varphi + f(u)\partial_1\varphi)\,dx \qquad \textit{(use Gauss theorem)} \\
&= -\int_{\Omega} (\partial_2 u + \partial_1(f(u)))\,\varphi dx.
\end{aligned}$$

From the arbitrariness of φ and Lemma 1.3.9, it follows $\partial_2 u + \partial_1(f(u)) = 0$ in Ω.

Now, let $x^0 \in \partial\Omega \cap \{x_2 = 0\}$ and $B(x^0, r)$ be a ball such that $B^{0,+} := B(x^0, r) \cap \{x_2 > 0\} \subset \Omega$, and denote by $\nu^{0,+}$ the exterior unit normal vector to $\partial B^{0,+}$. Taking $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\mathrm{supp}(\varphi) \subset B(x^0, r)$ and using Gauss theorem as above gives

$$\begin{aligned}
0 &= \int_{B^{0,+}} (u\partial_2\varphi + f(u)\partial_1\varphi))\, dx + \int_{\partial B^{0,+}\cap\{x_2=0\}} g(x_1)\varphi(x_1, 0)dx_1 \\
&= -\int_{B^{0,+}} (\partial_2 u + \partial_1(f(u)))\, \varphi dx \\
&+ \int_{\partial B^{0,+}\cap\{x_2>0\}} (u\nu_2^{0,+} + f(u)\nu_1^{0,+})\varphi ds \\
&+ \int_{\partial B^{0,+}\cap\{x_2=0\}} (g(x_1) + u(x_1, 0)\nu_2^{0,+} + f(u)\nu_1^{0,+})\varphi(x_1, 0)dx_1 \\
&= \int_{\partial B^{0,+}\cap\{x_2=0\}} (g(x_1) - u(x_1, 0))\varphi(x_1, 0)dx_1
\end{aligned}$$

which from the arbitrariness of x^0 and φ proves $u(\cdot, 0) = g(\cdot)$ on $\partial\Omega \cap \{x_2 = 0\}$ and completes the proof. □

Theorem 9.2.3 (Rankine-Hugoniot) *Let $\Omega \subset \mathbb{R} \times \mathbb{R}_+$ be a simply connected open set such that $\Omega = \Omega_l \cup \gamma \cup \Omega_r$, where γ is the graph of a function $x_1 = \chi(x_2)$, $x_2 \in (\alpha, \beta)$, $\gamma \in C^1(\alpha, \beta)$, and Ω_l, resp. Ω_r, is the open set at the left, resp. at the right, of γ; see Fig. 3.3.1. Assume that u is a weak solution to (3.3.1) in Ω such that $u \in C^1(\Omega_l) \cap C^0(\overline{\Omega}_l)$ and $u \in C^1(\Omega_r) \cap C^0(\overline{\Omega}_r)$, and set*

$$\begin{aligned}
[u] &= \lim_{h\to 0^+} (u(\chi(x_2) + h, x_2) - u(\chi(x_2) - h, x_2)), \\
[f(u)] &= \lim_{h\to 0^+} (f(u(\chi(x_2) + h, x_2)) - f(u(\chi(x_2) - h, x_2)) \;\; x_2 \in (\alpha, \beta).
\end{aligned}$$

Then

$$\chi'[u] = [f(u)] \quad \textit{on } \gamma. \tag{9.2.8}$$

Proof. WLOG we may assume that both Ω_l and Ω_r are smooth, for example Lipschitz, because otherwise we consider $\Omega_l \cap B(x, \rho)$, resp. $\Omega_r \cap B(x, \rho)$, instead of Ω_l, resp. Ω_r, with $x \in \gamma$ and ρ small enough. We will use Gauss theorem in Ω_l and Ω_r, and we denote by ν^l, resp. ν^r, the exterior unit normal vector to $\partial\Omega_l$, resp. $\partial\Omega_r$. As $x_1 - \chi(x_2) = 0$ is the equation of γ, it follows that $\nu^l = \dfrac{(1, -\chi')}{\sqrt{1 + (\chi')^2}} = -\nu^r$ on γ.

Next we note that as $u \in C^1(\Omega_l \cup \Omega_r)$ is a weak solution to (3.3.1) in $\Omega_l \cup \Omega_r$, from Proposition 3.3.4 it follows that u is a classical solution to (3.3.1) in Ω_l and Ω_r. Hence

$$\partial_2 u + \partial_1(f(u)) \;\; = \;\; 0 \quad \textit{in } \Omega_l \cup \Omega_r.$$

As u is a weak solution in Ω, for $\varphi \in \mathcal{D}(\Omega)$ we get

$$0 = \int_{\Omega_l} (u\partial_2\varphi + f(u)\partial_1\varphi)dx + \int_{\Omega_r} (u\partial_2\varphi + f(u)\partial_1\varphi)dx.$$

Now we use Gauss theorem for each of these integrals. By taking into account the fact that the integrals on $\partial\Omega_l \cap \partial\Omega$ and on $\partial\Omega_r \cap \partial\Omega$ vanish, we get

$$\begin{aligned}
0 &= \int_{\partial\Omega_l} \varphi(u\nu_2^l + f(u)\nu_1^l)ds + \int_{\partial\Omega_r} \varphi(u\nu_2^r + f(u)\nu_1^r)ds - \int_{\Omega_l\cup\Omega_r} \varphi(\partial_2 u + \partial_1(f(u)))dx \\
&= \int_\gamma \varphi(u\nu_2^l + f(u)\nu_1^l)ds + \int_\gamma \varphi(u\nu_2^r + f(u)\nu_1^r)ds \\
&= \int_\gamma \varphi([u]\nu_2^r + [f(u)]\nu_1^r)ds \\
&= \int_\gamma \frac{\varphi}{\sqrt{1+|\chi'|^2}}(-\chi'[u] + [f(u)])ds.
\end{aligned}$$

From the arbitrariness of φ, the last equality implies (9.2.8). □

9.3 Second-order linear elliptic PDEs: maximum principle and classical solutions (Ch. 4)

Throughout this chapter, $\Omega \subset \mathbb{R}^N$ is open bounded and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, unless otherwise specified.

9.3.1 Dirichlet problem in a ball

Theorem 9.3.1 *Let $R > 0$, $B_R := \{x \in \mathbb{R}^N,\ |x| < R\}$, V_N the volume of B_1, $g \in C^0(\partial B_R)$ and set*

$$u(x) = \frac{R^2 - |x|^2}{NV_N R} \int_{\partial B_R} \frac{g(y)}{|x-y|^N} d\sigma(y), \quad x \in B_R. \tag{9.3.1}$$

Then u is the unique $C^0(\overline{B}_R) \cap C^\infty(B_R)$ solution to (4.0.1) in $\Omega = B_R$.

Proof. Let $K : \partial B_R \times B_R \mapsto \mathbb{R}$ be defined by

$$K(x,y) = \frac{R^2 - |x|^2}{NV_N R} \frac{1}{|x-y|^N}. \tag{9.3.2}$$

Then $u(x) = \int_{\partial B_R} K(x,y) g(y) d\sigma(y)$. From direct calculus, one finds that $u \in C^\infty(B_R)$; see, for example, [28]. Furthermore, $\Delta u = 0$ in B_R because

$$\begin{aligned}
\Delta u(x) &= \sum_{i=1,N} \partial_{ii} u(x) \\
&= \partial_i \left(-\frac{2x_i}{NV_N R} \int_{\partial B_R} \frac{1}{|x-y|^N} g(y) d\sigma - \frac{R^2 - |x|^2}{V_N R} \int_{\partial B_R} \frac{(x-y)_i}{|x-y|^{N+2}} g(y) d\sigma \right) \\
&= -\frac{2}{V_N R} \int_{\partial B_R} \frac{1}{|x-y|^N} g(y) d\sigma + \frac{4}{V_N R} \int_{\partial B_R} \frac{x \cdot (x-y)}{|x-y|^{N+2}} g(y) d\sigma \\
&\quad - \frac{R^2 - |x|^2}{V_N R} \int_{\partial B_R} \frac{N}{|x-y|^{N+2}} g(y) d\sigma + \frac{R^2 - |x|^2}{V_N R} \int_{\partial B_R} \frac{N+2}{|x-y|^{N+2}} g(y) d\sigma \\
&= -\frac{2}{V_N R} \int_{\partial B_R} \frac{1}{|x-y|^N} g(y) d\sigma + \frac{2}{V_N R} \int_{\partial B_R} \frac{2x \cdot (x-y) + R^2 - |x|^2}{|x-y|^{N+2}} g(y) d\sigma \\
&= 0,
\end{aligned}$$

because $y \cdot y = R^2$ and so $2x \cdot (x-y) + R^2 - |x|^2 = x \cdot x - 2x \cdot y + y \cdot y) = |x-y|^2$.

To prove $u \in C^0(\overline{B}_R)$, we introduce the auxiliary function

$$u_1(x) = \int_{\partial B_R} K(x,y) d\sigma, \quad x \in B_R, \tag{9.3.3}$$

which is obtained from (9.3.1) for $g = 1$. Thus $u_1 \in C^\infty(B_R)$ and $\Delta u_1 = 0$ in B_R. Furthermore, it is easy to check that u_1 is radially symmetric, and, then, from the formula of Δu in spherical coordinates[3], we have $\Delta u_1 = \frac{1}{r^{N-1}}\partial_r\left(r^{N-1}\partial_r u_1\right) = 0$. Therefore, u_1 is constant in B_R and as $u_1(0) = \int_{\partial B_R} \frac{R^2}{NV_N R}\frac{1}{R^N}d\sigma(y) = 1$ it follows that $u_1 = 1$ in B_R.

Now, let us prove $u \in C^0(\overline{B}_R)$. Let $x_0 \in \partial B_R$ and $\epsilon > 0$. We have to find $\delta > 0$ such that

$$\forall x \in B(x_0,\delta)\cap B_R, \quad |u(x) - g(x_0)| < \epsilon. \tag{9.3.4}$$

From $g \in C^0(\partial B_R)$, there exists $\delta_1 > 0$ such that for all $y, z \in \partial B_R$ with $|y - z| < 2\delta_1$ we have $|g(y) - g(z)| < \frac{\epsilon}{2}$. It follows that for $x \in B_R \cap B(x_0, \delta_1)$, we have

$$\begin{aligned}
|u(x) - g(x_0)| &= |u(x) - g(x_0)u_1(x)| \\
&= \left|\int_{\partial B_R} K(x,y)(g(y) - g(x_0))d\sigma(y)\right| \\
&\le \int_{\partial B_R\cap\{|y-x_0|<2\delta_1\}} K(x,y)|g(y)-g(x_0)|d\sigma(y) + \int_{\partial B_R\cap\{|y-x_0|\ge 2\delta_1\}} K(x,y)|g(y)-g(x_0)|d\sigma(y) \\
&\le \frac{\epsilon}{2}\int_{\partial B_R} K(x,y)d\sigma(y) + 2\|g\|_{C^0(\partial B_R)}\frac{R^2-|x|^2}{NV_N R\delta_1^N}\int_{\partial B_R} d\sigma(y) \\
&\le \frac{\epsilon}{2} + 2\|g\|_{C^0(\partial B_R)}\frac{|\partial B_R|}{NV_N R\delta_1^N}(R^2 - |x|^2).
\end{aligned}$$

As $0 \le R^2 - |x|^2 \le 2R|x - x_0|$, there exists $\delta_2 > 0$ such that for all $x \in B_R \cap B(x_0, \delta_2)$ we have

$$2\|g\|_{C^0(\partial B_R)}\frac{|\partial B_R|}{NV_N R\delta_1^N}(R^2 - |x|^2) < \frac{\epsilon}{2}.$$

Then (9.3.4) is satisfied with $\delta = \min\{\delta_1, \delta_2\}$ and so $u \in C^0(\overline{B}_R)$. Finally, the uniqueness of the solution to (4.0.1) with $\Omega = B_R$ follows from (ii), Corollary 4.3.2. □

9.3.2 Maximum principle for second-order linear elliptic PDEs

Throughout this section, we will consider

$$Lu = -\sum_{i,j=1}^N a_{ij}\partial_{ij}u + \sum_{i=1}^N b_i\partial_i u + cu, \quad A = (a_{ij}), \quad b = (b_i), \tag{9.3.5}$$

[3] If $(r,\theta) \in (0,\infty)\times\mathbb{S}^{N-1}$ are the spherical coordinates connected to the Cartesian coordinates x by $x = r\theta$, $\theta = \frac{x}{|x|}$, then the classical formula holds: $\Delta_x u(x) = \frac{1}{r^{N-1}}\partial_r(r^{N-1}\partial_r u(r\theta)) + \Delta_x u\left(r\frac{x}{|x|}\right)$.

and we will assume the following:[4]

$$\Omega \subset \mathbb{R}^N \textit{ is an open set}, \quad u \in C^0(\overline{\Omega}) \cap C^2(\Omega), \tag{9.3.6a}$$

$$a_{ij},\ b_i,\ c \in C^0(\overline{\Omega}) \ \textit{ for all } i, j, \tag{9.3.6b}$$

$$A(x) \textit{ is symmetric for all } x \in \overline{\Omega} \ \textit{ and } \ A(x) > 0 \ \textit{ in } \overline{\Omega}. \tag{9.3.6c}$$

Theorem 9.3.2 (weak maximum principle) *Assume $\Omega \subset \mathbb{R}^N$ is bounded and $c = 0$. If $Lu \le 0$, resp. $Lu \ge 0$, then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \qquad \textit{resp.} \quad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u,$$

where $\min_A f := \min\{f(x),\ x \in A\}$, $\max_A f := \max\{f(x),\ x \in A\}$.

Proof. We deal first with the case $Lu \le 0$. Let $x_0 \in \overline{\Omega}$ such that $u(x_0) = \max\{u(x),\ x \in \overline{\Omega}\}$. We distinguish two sub-cases.
(i) Case $Lu < 0$ in Ω. If $x^0 \in \partial\Omega$ the theorem is proved. So we assume $x^0 \in \Omega$. We have $|\nabla u(x^0)| = 0$ and $D^2u(x^0) = [\partial_{ij}u(x^0)] \le 0$. Therefore

$$Lu(x^0) \ = \ -\sum_{i,j=1}^{N} a_{ij}(x^0)\partial_{ij}u(x^0) = -\mathrm{tr}\left[A(x^0) \cdot D^2u(x^0)\right] \ge 0,$$

because from $A(x^0) > 0$ and $D^2u(x^0) \le 0$ it follows $\mathrm{tr}[A(x^0) \cdot D^2u(x^0)] \le 0$,[5] which is a contradiction and proves that $x^0 \in \partial\Omega$.
(ii) Case $Lu \le 0$ in Ω. Let $\epsilon,\ \gamma > 0$ and consider $u_\epsilon = u + \epsilon e^{\gamma x_1}$. We have

$$Lu_\epsilon = Lu + \epsilon L e^{\gamma x_1} = Lu + \epsilon(-\gamma^2 a_{11} + \gamma b_1)e^{\gamma x_1}.$$

We can choose γ large enough such that $(-\gamma^2 a_{11} + \gamma b_1)e^{\gamma x_1} < 0$ in $\overline{\Omega}$. This is possible because Ω is bounded, $a_{11} > 0$ (this follows from $A > 0$), and (9.3.6b). So $Lu_\epsilon < 0$ in Ω. From case *(i)*, it follows that there exists $x^\epsilon \in \partial\Omega$ such that

$$u_\epsilon(x) \le u_\epsilon(x^\epsilon), \quad \textit{so} \quad u(x) \le u(x^\epsilon) + \epsilon(e^{\gamma x_1^\epsilon} - e^{\gamma x_1}), \quad x \in \overline{\Omega}.$$

There exists a subsequence of (x^ϵ), still denoted by (x^ϵ), and $x^0 \in \partial\Omega$ such that $\lim_{\epsilon\to 0} x^\epsilon = x^0$, which implies that for all $x \in \overline{\Omega}$ we have $u(x) \le u(x^0)$.

[4] A matrix $M \in \mathbb{R}^{N\times N}$ is said to be positive, resp. strictly positive, and we write $M \ge 0$, resp. $M > 0$, if $\xi \cdot M \cdot \xi \ge 0$, resp. $\xi \cdot M \cdot \xi > 0$, for every $0 \ne \xi \in \mathbb{R}^N$.

[5] Indeed, let $A(x^0) = U \cdot \Lambda \cdot {}^tU$ be the spectral decomposition of $A(x^0)$ with $U^{-1} = {}^tU$ and Λ the diagonal matrix of eigenvalues of $A(x^0)$. From the cyclic property of the trace, we get $\mathrm{tr}[A(x^0) \cdot D^2u(x^0)] = \mathrm{tr}[\Lambda \cdot {}^tU \cdot D^2u(x^0) \cdot U)] = \mathrm{tr}[\Lambda \cdot B]$, with $B = {}^tU \cdot D^2u(x^0) \cdot U$; clearly $B \le 0$, because $D^2u(x^0) \le 0$, and then $\mathrm{tr}[A(x^0) \cdot D^2u(x^0)] = \mathrm{tr}[\Lambda \cdot B] = \sum_i \lambda_i B_{i,i} \le 0$ because $\lambda_i > 0$ and $B_{i,i} \le 0$ for all i.

For the case $Lu \geq 0$ we set $v = -u$, so $Lv \leq 0$, and then the result follows from *(ii)*. $\square$

Note that this theorem does not hold if $c \neq 0$. For example, if $\Omega = B(0,1) \subset \mathbb{R}^2$ and $Lu = -\Delta u - u$ (so $c = -1$) then for $u = 5 - |x|^2$, we have

$$Lu = -1 + |x|^2 \leq 0 \ \ \textit{in } \Omega, \quad \textit{and} \ \ \max_{\overline{\Omega}} u = 5 > 4 = \max_{\partial\Omega} u.$$

However, if $c \geq 0$ we have the following result.

Corollary 9.3.3 *Assume Ω bounded and $c \geq 0$. If $Lu \leq 0$, resp. $Lu \geq 0$, in Ω then*

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+, \qquad \textit{resp.} \quad \min_{\partial\Omega} u^- \leq \min_{\overline{\Omega}} u, \tag{9.3.7}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$.

Proof. Let us consider first the case $Lu \leq 0$. Set $\Omega^+ := \{x \in \Omega, \ u(x) > 0\}$. As $u \in C^0(\overline{\Omega})$ it follows that Ω^+ is open. There are two possibilities.
(i) $\Omega^+ = \emptyset$. So $u \leq 0$ in Ω and then the claim is proved.
(ii) $\Omega^+ \neq \emptyset$. Then $cu \geq 0$ in Ω^+. If we set $L^+u := -\sum_{i,j=1}^N a_{ij}\partial_{ij}u + \sum_{i=1}^N b_i\partial_i u + 0u$, we have $L^+u \leq 0$ in Ω^+. Applying Theorem 9.3.2 with L^+ in Ω^+ (instead of L in Ω) implies $\max_{\overline{\Omega^+}} u^+ = \max_{\partial(\Omega^+)} u^+$.

Note that necessarily $\partial(\Omega^+) \cap \partial\Omega \neq \emptyset$, because otherwise $\partial\Omega^+ \subset \Omega$ and then $\max_{\overline{\Omega^+}} u^+ = \max_{\partial(\Omega^+)} u^+ = 0$, which is impossible because $\Omega^+ \neq \emptyset$. Then we get

$$\max_{\overline{\Omega}} u = \max_{\overline{\Omega^+}} u = \max_{\overline{\Omega^+}} u^+ = \max_{\partial(\Omega^+)} u^+ = \max_{\partial\Omega} u^+,$$

which proves the corollary in the case $Lu \leq 0$.
(iii) The case $Lu \geq 0$ is proved by setting $v = -u$ and using $Lv \leq 0$. $\square$

Note that if $\max_{\overline{\Omega}} u < 0$ or $\min_{\overline{\Omega}} u < 0$, the result of the previous corollary is not optimal.

Corollary 9.3.4 (comparison principle) *Let Ω be open bounded and $c \geq 0$ in Ω.*

(i) If $Lu \leq Lv$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω (monotonicity of L).

(ii) If $Lu = Lv$ in Ω and $u = v$ on $\partial\Omega$, then $u = v$ in Ω (uniqueness of $Lu = 0$).

Proof. Set $w = u - v$.
(i) We have

$$Lw \leq 0, \ \ c \geq 0 \quad \textit{in} \ \ \Omega, \quad w \leq 0 \quad \textit{on} \ \ \partial\Omega.$$

From Corollary 9.3.3, it follows that $\max_{\overline{\Omega}} w \leq \max_{\partial\Omega} w^+ \leq 0$. So, $w \leq 0$, or $u \leq v$.
(ii) As $Lw = 0$ and $w = 0$, (i) implies $w = 0$, so $u = v$.

Lemma 9.3.5 (Hopf lemma) *Let $\Omega \subset \mathbb{R}^N$ be an open set with $\partial\Omega$ a C^2 boundary near*[6] *$x_0 \in \partial\Omega$, $Lu(x) \leq 0$, $u(x) < u(x_0)$ for all $x \in \Omega$ near x_0 and u differentiable at x_0. Furthermore, assume that one of the following holds:*
(i) $c = 0$ *near* x_0,
(ii) $u(x_0) \geq 0$ *and* $c \geq 0$ *near* x_0,
(iii) $u(x_0) = 0$.
Then $\partial_\nu u(x_0) := \nabla u(x_0) \cdot \nu(x_0) > 0$, where $\nu(x_0)$ is the unit outward normal vector to $\partial\Omega$ at x_0.

Proof. From $u(x_0) - u(x) \geq 0$ near x_0 in Ω, we obtain $\partial_\nu u(x_0) \geq 0$. The lemma would be proven if we find a function $v \in C^1(\overline{\Omega})$ such that $u(x_0) - u(x) \geq v(x_0) - v(x)$ near x_0 and $\partial_\nu v(x_0) > 0$.
Construction of v. Let $z \in \Omega$ and $r > 0$ such that if $B_r(z)$ is the ball with center z and radius r, then $B_r(z) \subset \Omega$, $\partial B_r(z) \cap \partial\Omega = \{x_0\}$; see Figure 9.3.1. Set $V = B_r(z) \cap B_{r_0}(x_0)$, with r_0 small, and $v(x) = \epsilon(e^{-k|x_0-z|^2} - e^{-k|x-z|^2})$, with $x \in V$, $\epsilon, k > 0$. Note that

$$\begin{aligned} v(x_0) - v &\geq 0 \quad \text{in } V, \\ L(v(x_0) - v) &= \epsilon\Big(\big(-4k^2(x-z)\cdot A(x)\cdot(x-z) + 2k(\mathrm{tr}(A(x)) - (x-z)\cdot b\big)e^{-k|x-z|^2} \\ &\qquad -c\,(e^{-k|x_0-z|^2} - e^{-k|x-z|^2})\Big), \\ L(u(x_0) - u) &\geq cu(x_0). \end{aligned}$$

With k large, ϵ small, and with one of (i)–(iii) holding, we obtain

$$\begin{aligned} v(x_0) - v \ \leq\ & u(x_0) - u && \text{on } \partial V, \\ L(v(x_0) - v) \leq 0 \ \leq\ & L(u(x_0) - u) && \text{in } V, \end{aligned}$$

because $v(x_0) - v = 0$ on $\overline{V} \cap \partial B_r(z)$ and $u(x_0) - u > 0$ on $\overline{V} \cap \partial B_{r_0}(x_0)$, which is compact.
Cases (i) and (ii). As $c \geq 0$ in V, by applying Corollary 9.3.4 we get $v(x_0) - v(x) \leq u(x_0) - u(x)$ in V. Therefore, if we take $x = x_0 - t\nu(x_0)$, where $t > 0$ small, we get $v(x_0) - v(x_0 - t\nu(x_0)) \leq u(x_0) - u(x_0 - t\nu(x_0))$, which after dividing by t and letting t to zero gives $0 < 2kr\epsilon e^{-kr^2} \leq \partial_\nu u(x_0)$.

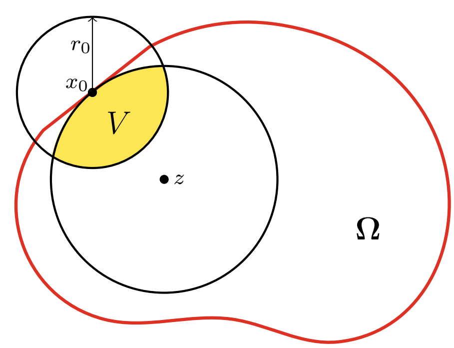

Figure 9.3.1: V and x_0

Case (iii). As $u(x_0) = 0$, we have $u < 0$ in Ω near x_0. Set $c^- = \min\{c, 0\}$. It follows that for L^- defined by $L^-u := -\sum_{i,j=1}^N a_{ij}\partial_{ij}u + \sum_{i=1}^N b_i\partial_i u + (c - c^-)u$, we have $L^-u = Lu - c^-u \leq Lu \leq 0$ in Ω. Then we can apply *Case (ii)* above with $L^-u \leq 0$, $u(x_0) = 0$ and $(c - c^-) \geq 0$, which completes the proof. □

[6] Note that this implies Ω lies on one side of $\partial\Omega$.

Remark 9.3.6 *In the case of a local minimum, Hopf lemma is stated as follows. Let $\Omega \subset \mathbb{R}^N$ be open with $\partial\Omega$ a C^2 boundary near $x_0 \in \partial\Omega$, $Lu(x) \geq 0$, $u(x) > u(x_0)$ for all $x \in \Omega$ near x_0, and u differentiable at x_0. Assume also that one of the following holds:*
(i) $c = 0$ *near* x_0,
(ii) $u(x_0) \leq 0$ *and* $c \geq 0$ *near* x_0,
(iii) $u(x_0) = 0$.
Then $\partial_\nu u(x_0) < 0$. The proof follows from Lemma 9.3.5 applied to $-u$.

As a consequence of Lemma 9.3.5, we have the following theorem.

Theorem 9.3.7 (Strong maximum principle) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume $Lu \leq 0$, resp. $Lu \geq 0$, in Ω. Then u is constant or, if not, we have*
(i) if $c = 0$ then u does not achieve its maximum, resp. minimum, in Ω,
(ii) if $c \geq 0$ then u does not achieve a non-negative maximum, resp. non-positive minimum, in Ω,
(iii) regardless of the sign of c, u does not achieve a maximum, resp. minimum, in Ω equal to zero.

Proof. Let us consider first the case $Lu \leq 0$. Let $M = \max\{u(x),\, x \in \overline{\Omega}\}$ and set $K = \{x \in \overline{\Omega},\, u(x) = M\}$. Note that K is closed. If $K = \overline{\Omega}$ then $u = M$ in Ω and so the theorem is proved. Otherwise, we assume that one of (i)–(iii) fails. Each case implies that u achieves its maximum value M in Ω. So, $K \cap \Omega \subsetneq \Omega$.

As in Theorem 4.3.4, there exists $x_0 \in \partial K$, $B_0 := B(z_0, r_0) \Subset \Omega \backslash K$ (see Fig. 4.3.1), such that u achieves at x_0 a strict maximum in $\overline{B}_0$, and so $|\nabla u(x_0)| = 0$. Note that u satisfies the conditions of Lemma 9.3.5 at x_0, with B_0 instead of Ω. Applying Lemma 9.3.5 implies $|\nabla u(x_0)| > 0$. The contradiction shows that one of (i)–(iii) holds.

The result for the case $Lu \geq 0$ is obtained by setting $v = -u$ and using the case $Lu \leq 0$.

9.3.3 Solution to the Dirichlet problem

9.3.3.1 Sub(super) harmonic functions and sub(super) solutions

Proposition 9.3.8 *Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that $-\Delta u = 0$ in Ω.*

i) u is a subharmonic and superharmonic function.

ii) Assume u solves (4.0.1). Then u is a subsolution and supersolution to (4.0.1).

Proof. Let B be a ball, $B \Subset \Omega$, $h \in C^0(\overline{B}) \cap C^2(B)$, $-\Delta h = 0$ in B, and $u \leq h$ on ∂B. Up to a translation, we may assume that $B = B_R$, for a certain $R > 0$. Then from (4.2.1), for $x \in B_R$ we have

$$(u - h)(x) = \frac{R^2 - |x|^2}{NV_N R} \int_{\partial B_R} \frac{u(y) - h(y)}{|x - y|^N} d\sigma(y) \leq 0.$$

So u is a subharmonic function in Ω. Similarly, we prove that u is a superharmonic. The Claim ii) is a corollary of i).

Proposition 9.3.9 *Let $u, v \in C^0(\overline{\Omega})$.*

i) Assume u is subharmonic and v is superharmonic in Ω with $u \le v$ on $\partial\Omega$. Then either $u < v$ or $u = v$ in Ω.

ii) Assume u is a subsolution and v is a supersolution to (4.0.1). Then either $u < v$ or $u = v$ in Ω.

Proof. Let us prove first Claim i). Let $M = \max\{u(x) - v(x),\ x \in \overline{\Omega}\}$, which exists as $u, v \in C^0(\overline{\Omega})$. If $M < 0$ then Claim i) holds. So it remains the case $M \ge 0$. If $u - v \equiv M$ in Ω, the case $M = 0$ gives $u = v$ in Ω, which again proves i), while the case $M > 0$ is impossible because it gives $u = M + v > v$ on $\partial\Omega$, which is a contradiction.

So it remains the case that $M \ge 0$ and $u - v \lneqq M$ in Ω. Set $K = \{x \in \overline{\Omega},\ (u-v)(x) = M\}$. The set K is nonempty and closed and $\Omega\backslash K$ is nonempty and open; see Figure 9.3.2. Consequently,

$$\exists x_0 \in \partial K \cap \Omega,\ \exists r_0 > 0 \ \text{ such that } \ B_0 := B(x_0, r_0) \Subset \Omega.$$

Now consider $\overline{u}, \overline{v} \in C^0(\overline{B}_0)$ defined by

$$\begin{cases} \Delta\overline{u} = 0 & \text{in } B_0, \\ \overline{u} = u & \text{on } \partial B_0, \end{cases} \qquad \begin{cases} \Delta\overline{v} = 0 & \text{in } B_0, \\ \overline{v} = v & \text{on } \partial B_0. \end{cases}$$

Then

$$\begin{aligned} M &= u(x_0) - v(x_0) \\ &\le (\overline{u} - \overline{v})(x_0) = \frac{1}{|\partial B_0|} \int_{\partial B_0} (\overline{u} - \overline{v})(y) d\sigma(y) \\ &< M, \end{aligned}$$

because $u - v \lneqq M$ on ∂B_0. The contradiction shows that this case is impossible and proves i). The Claim ii) follows from i).

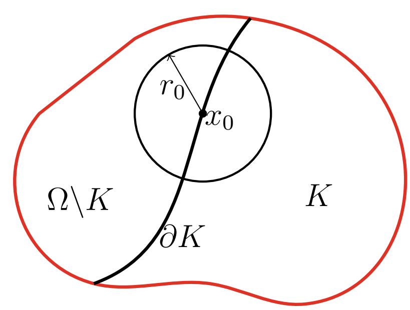

Figure 9.3.2: K, $\Omega\backslash K$, and $B(x_0, r_0)$

Proposition 9.3.10 *Let u be subharmonic in Ω, B a ball with $B \Subset \Omega$, and $\overline{u} \in C^0(\overline{B}) \cap C^2(B)$ such that $-\Delta\overline{u} = 0$ in B and $\overline{u} = u$ on ∂B. Then U, given by*

$$U(x) = \overline{u}(x), \ \ x \in B, \qquad U(x) = u(x), \ \ x \in \Omega\backslash B,$$

called "harmonic lifting of u with respect to B", is also subharmonic in Ω.

Proof. Let C be a ball, $C \Subset \Omega$, and $h \in C^0(\overline{C}) \cap C^2(C)$ satisfying $-\Delta h = 0$ in C, $U \leq h$ on ∂C. To prove is $U \leq h$ in C.

If $C \cap B = \emptyset$ then $U \leq h$ in C is equivalent to $u \leq h$ in C, which is true because u is subharmonic in Ω. Similarly, if $C \subset B$ the proposition is also true because $U \leq h$ in C is equivalent to $\overline{u} \leq h$ in C, which again is true because $\overline{u}$ is harmonic in C (see Proposition 4.4.2 with $\Omega = B$).

It remains the case that $C \cap B \neq \emptyset$, $C \cap (\Omega \setminus B) \neq \emptyset$; see Figure 9.3.3. Note that $u \leq U$ in Ω and $U \leq h$ in ∂C, so it follows $u \leq h$ in ∂C. Therefore, as u is a subsolution, we have $u \leq h$ in C. From $u = U$ in $C \setminus B$, it follows

$$U \leq h \quad in \quad C \setminus B.$$

Furthermore, we have

$$\begin{array}{rcll} -\Delta U = & 0 & = -\Delta h & in\ C \cap B, \\ U & \leq & h & on\ \partial(C \cap B) \cap \overline{B}, \\ & & \textit{(from the assumption for } h\textit{)} & \\ U & \leq & h & on\ \partial(C \cap B) \cap \overline{C}. \\ & & \textit{(from } U \leq h \textit{ in } C \setminus B\textit{)} & \end{array}$$

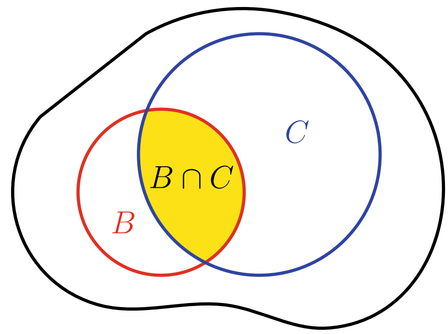

Figure 9.3.3: Harmonic lifting

Then from (ii), Corollary 4.3.2, it follows $U \leq h$ in $C \cap B$, so $U \leq h$ in C and this completes the proof.

Proposition 9.3.11 *If $u_1, \ldots, u_m$ are subharmonic functions in Ω, then* $\max\{u_1, \ldots, u_m\}$ *is also subharmonic in Ω.*

Proof. Define $\overline{u}$ by $\overline{u}(x) = \max\{u_1(x), \ldots, u_m(x)\}$, $x \in \overline{\Omega}$. It is easy to show that $\overline{u} \in C^0(\overline{\Omega})$. Let B be a ball with $B \Subset \Omega$ and $h \in C^0(\overline{B}) \cap C^2(\Omega)$ satisfying $-\Delta h = 0$ and $\overline{u} \leq h$ on ∂B. We have $u_i \leq \overline{u} \leq h$ on ∂B for all i. Therefore, $u_i \leq h$ in B, for all i, which implies $\overline{u} \leq h$ in B.

9.3.3.2 Some auxiliary results from Analysis

To conclude the existence of solutions to (4.0.1), we need a compactness result in $C^0(\overline{B})$, with $B \subset \mathbb{R}^N$ a ball, for bounded harmonic functions. This result relies on the concept of uniform equicontinuity of sequences given by this definition.

Definition 9.3.12 *Let $K \subset \mathbb{R}^N$ and $f_n : K \mapsto \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions.*

i) The sequence (f_n) is called "equicontinuous at $x \in K$" if

$$\forall \epsilon > 0,\ \exists \delta = \delta_x > 0 \ \ s.th.\ \forall y \in K \ \ with\ |x - y| < \delta \ \ and\ \forall n \in \mathbb{N} : \\ |f_n(x) - f_n(y)| < \epsilon.$$

ii) We say (f_n) is equicontinuous in K if (f_n) is equicontinuous at every $x \in K$.

iii) We say (f_n) is uniformly equicontinuous in K if (f_n) is equicontinuous in K with δ independent of x.

iv) Also (an independent concept from equicontinuity), we say (f_n) is bounded in K if there exists $M \geq 0$ such that $|f_n(x)| \leq M$ for all $x \in K$ and n.

Example 9.3.13 *Let (f_n) be a uniformly Lipschitz sequence in $C^0(K)$, i.e. $f_n \in C^0(K)$, for all n, and*

$$\exists L > 0, \quad |f_n(x) - f_n(y)| \leq L|x - y|, \quad \forall x, y \in K, \ \forall n \in \mathbb{N}.$$

Then (f_n) is uniformly equicontinuous in K, because for $|x - y| < \delta := \epsilon/L$ we have $|f_n(x) - f_n(y)| < \epsilon$ for all n.

More generally, if (f_n) is equicontinuous in K and K is compact, then (f_n) is uniformly equicontinuous. Indeed, for given $\epsilon > 0$ let δ_x be as in i), Definition 9.3.12. From the open covering $\{B(x, \delta_x/2), \ x \in K\}$ of K, there exists a finite covering $\{B(x_i, \delta_i/2), \ i = 1, \dots, m\}$ of K. If $\delta = \min\{\delta_i/2, \ i = 1, \dots, m\}$ and $x, y \in K$ with $|x - y| < \delta$, then necessarily $x, y \in B(x_j, \delta_j)$, for a certain j, and then i) in Definition 9.3.12 is satisfied with this δ, and for all x, y and n.

The following result states that an equicontinuous uniformly bounded sequence is compact in C^0. For its proof see any Real Analysis classic book, for example [28].

Theorem 9.3.14 (Arzela-Ascoli lemma) *Let $K \subset \mathbb{R}^N$ be a compact set and (f_n) be a equicontinuous and uniformly bounded sequence in K. Then (f_n) has a convergent subsequence in $C^0(K)$.*

The following result shows that a bounded sequence of harmonic functions is compact in a certain sense. It is used in the proof of Theorem 4.4.7 in a substantial way to show that the maximum of subsolutions to (4.0.1) is a harmonic function; see the comment after Proposition 4.4.5.

Theorem 9.3.15 *Let $\Omega \subset \mathbb{R}^N$ be an open set and (u_n) be a sequence of functions, harmonic and uniformly bounded in Ω by a constant M. Then (u_n) has a subsequence (u_{n_k}) converging pointwise in Ω to a harmonic function u in Ω. The convergence is in $C^0(K)$ for every compact $K \subset \Omega$.*

Proof. For $m \in \mathbb{N}$, set $K_m = \left\{x \in \Omega, \ d(x, \partial\Omega) \geq \dfrac{2}{m}, \ |x| \leq m\right\}$, where $d(x, \partial\Omega) = \inf\{|x - y|, \ y \in \partial\Omega\}$ is the distance of x to $\partial\Omega$. Then K_m is compact and $\Omega = \bigcup_{m \in \mathbb{N}} K_m$.

For $x \in K_m$ we have $B_m := B(x, \frac{1}{m}) \subset \overline{B}_m \subset \Omega$. Poisson's formula (4.2.2) for u_n in B_m gives

$$u_n(z) = \frac{R_m^2 - |z - x|^2}{N V_N R_m} \int_{\partial B_m} \frac{u_n(y)}{|z - y|^N} d\sigma(y), \quad z \in B_m, \quad R_m := \frac{1}{m}.$$

As $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded, this formula gives a uniform bound for ∇u_n in K_m. Indeed, differentiating $u_n(z)$ at $z=x$ gives

$$|\nabla u_n(x)| \leq \frac{2|x-x|}{NV_N R_m}\int_{\partial B_m}\frac{M}{R_m^N}d\sigma(y)+\frac{R_m^2}{V_N R_m}\int_{\partial B_m}\frac{M}{R_m^{N+1}}d\sigma(y)=:C(m), \tag{9.3.8}$$

with a certain $C(m)>0$.

It follows that $(u_n)_{n\in\mathbb{N}}$ satisfies the conditions of Theorem 9.3.14 in K_m (because the bound (9.3.8) implies that (u_n) is a uniformly Lipschitz sequence; see Example 9.3.13). Therefore, there exists a subsequence $(u_n^m)_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ and $u^m\in C^0(K_m)$ such that $\lim\limits_{n\to\infty} u_n^m = u^m$ in $C^0(K_m)$.

We set $(u_n^0)=(u_n)$. From the reasoning above, we obtain $(u_n^1)_{n\in\mathbb{N}}$, a subsequence of $(u_n^0)_{n\in\mathbb{N}}$, and $v^1\in C^0(K_1)$ such that $\lim\limits_{n\to\infty} u_n^1 = v^1$ in $C^0(K_1)$. Next we proceed with $(u_n^m)_{n\in\mathbb{N}}$ as with $(u_n^{m-1})_{n\in\mathbb{N}}$, $m=1,2,\ldots$. As a result of this process, we obtain the sequences $(u_n^m)_{n\in\mathbb{N}}$, and $v^m\in C^0(K_m)$, $m=1,2,\ldots$, such that $\lim\limits_{n\to\infty} u_n^m = v^m$ in $C^0(K_m)$.

Set $u^n=u_n^n$ for all $n\in\mathbb{N}$, and $v(x)=\lim_{n\to\infty} v^n(x)$. Note that $v(x)$ is well-defined in Ω, because $x\in K_m$, for a certain m, and from the construction, we have that up to a finite number of terms, $(u^n)_{n\in\mathbb{N}}$ is a subsequence of $(u_n^m)_{n\in\mathbb{N}}$.

The sequence (u^n) and v satisfies the theorem. Indeed, from the definition of (u^n) we have $\lim\limits_{n\to\infty} u^n = v$ in $C^0(K_m)$, because (u^n) is (up to a finite number of terms) a subsequence of $(u_n^m)_{n\in\mathbb{N}}$, for every m.

It remains to prove that u is harmonic in Ω. Let B be a ball with radius R, $B\Subset\Omega$. Then $u^n(z)=\dfrac{R^2-|z-x|^2}{NV_N R}\displaystyle\int_{\partial B}\frac{u^n(y)}{|z-y|^N}d\sigma$, $z(y)\in B$. Passing to the limit $n\to\infty$ inside the integral is allowed, because the convergence of u^n is uniform in $\overline{B}$ ($B\subset K_m$ for m large). This implies $u(z)=\dfrac{R^2-|z-x|^2}{NV_N R}\displaystyle\int_{\partial B}\frac{u(y)}{|z-y|^N}d\sigma(y)$ and then from Proposition 4.2.4 we obtain $-\Delta u=0$ in Ω.

9.3.3.3 Proof of Theorem 4.4.7

Set $\mathcal{S}_g=\{v:\Omega\mapsto\mathbb{R},\ v \textit{ subsolution to problem } (4.0.1)\}$ and $u(x)=\sup\{v(x),\ v\in\mathcal{S}_g\}$. The function u is well-defined. Indeed, there exist subsolutions and supersolutions to (4.0.1), for example $\underline{v}:=-\|g\|_{C^0(\partial\Omega)}$ is a subsolution and $\overline{v}:=\|g\|_{C^0(\partial\Omega)}$ is a supersolution. Then, $\mathcal{S}_g$ is not empty and from Corollary 4.4.3 for all $v\in\mathcal{S}_g$ we have $v\leq\overline{v}$, which proves that u is well-defined. Note also that, without restriction, we may assume that (v_n) is uniformly bounded in Ω.

Claim: The function u is harmonic in Ω.

Proof. Let $x\in\Omega$ and (v_n) in $\mathcal{S}_g$ with $\lim\limits_{n\to\infty} v_n(x)=u(x)$. Let $B=B(x,R)$ with $R>0$ and $B\Subset\Omega$, and consider V_n, the harmonic lifting of v_n w.r.t. B. As $v_n\leq V_n$ and V_n is a subsolution, we get $u(x)=\lim\limits_{n\to\infty} V_n(x)$. From Theorem 4.4.6 there exists a subsequence

of (V_n), still denoted by (V_n), converging pointwise in B to a harmonic function V in B, and converging uniformly in any compact subset of B.

Of course $V \leq u$ in B. In fact we have $V = u$ in B. Indeed, assume for the moment that $V(y) < u(y)$ for a certain $y \in B$. Then there exists $w \in S_g$ such that $V(y) < w(y)$. Set $w_n = \max\{w, V_n\}$ and let W_n be the harmonic lifting of w_n w.r.t. B. As for (V_n), a certain subsequence of (W_n), still denoted by (W_n), converges pointwise in B and uniformly in any compact subset of B to a certain harmonic function W in B. As $\max\{w, V_n\} \leq W_n$ in B, it follows

$$V \leq W \ \textit{ in } B, \quad V(y) < w(y) \leq W(y) \ \textit{ and } \ V(x) = W(x) = u(x).$$

From the strong maximum principle, Theorem 4.3.4, $V - W$ must reach the maximum on ∂B (and not in B) or, otherwise it must be constant. Therefore, as $(V - W)(x) = 0$ and $V - W \leq 0$ in B, it follows $V - W = 0$ in B, which contradicts $V(y) - W(y) < 0$. So $V = u$ in B and therefore u is harmonic in Ω. Theorem 4.2.1 implies that $u \in C^\infty(\Omega)$.
Claim: $u \in C^0(\overline{\Omega})$ and $u = g$ on $\partial\Omega$.
Proof. It is enough to prove that u is continuous at every $\xi \in \partial\Omega$ and $u = g$ on $\partial\Omega$. Let $\xi \in \partial\Omega$, $y \in \overline{\Omega}^c$, $R > 0$, and $B = B(y, R)$ such that $B \cap \Omega = \emptyset$, $\partial B \cap \partial\Omega = \xi$; see Figure 4.4.1. The choice of ξ and B is possible because $\partial\Omega$ is C^2. Now consider the function w given by

$$\begin{aligned} w(x) &= \ln \frac{|x-y|}{R}, & \textit{for} \quad N = 2, \\ w(x) &= R^{2-N} - |x-y|^{2-N}, & \textit{for} \quad N \geq 3, \end{aligned}$$

which satisfies

$$w \in C^\infty(\mathbb{R}^N \backslash \{y\}), \quad -\Delta w = 0 \ \textit{ in } \Omega, \quad w > 0 \ \textit{ in } \overline{\Omega} \backslash \{\xi\}, \quad w(\xi) = 0. \tag{9.3.9}$$

Set $v^-(x) = g(\xi) - (\epsilon + kw(x))$, resp. $v^+(x) = g(\xi) + (\epsilon + kw(x))$, with $\epsilon > 0$ given. Clearly,

$$-\Delta v^- = -\Delta v^+ = 0 \qquad \textit{in } \Omega.$$

Assume for a moment that $v^- \leq g \leq v^+$ on $\partial\Omega$, i.e. $v^-(x)$, resp. $v^+(x)$, is a subsolution, resp. supersolution, to (4.0.1). Then from Corollary 4.4.3 and the fact that $u(x) = \sup\{v(x),\, v \in \mathcal{S}_g\}$, we get

$$g(\xi) - (\epsilon + kw(x)) = v^-(x) \leq u(x) \leq v^+(x) = g(\xi) + (\epsilon + kw(x)).$$

This proves $|u(x) - g(\xi)| \leq \epsilon + kw(x)$ for all $x \in \overline{\Omega}$, which implies $\lim\limits_{x \to \xi} u(x) = g(x)$ because w is continuous and $w(\xi) = 0$.

Now we show that for an appropriate choice of ϵ, δ, the function v^-, resp. v^+, is a subsolution, resp. supersolution. Let $M := \|g\|_{C^0(\partial\Omega)}$ and choose ϵ, δ, k positive such that

$$|g(x) - g(\xi)| < \epsilon, \quad \textit{if} \quad x \in \partial\Omega \cap B(x, \delta),$$

$$kw(x) \geq 2M, \quad \textit{if} \quad x \in \Omega \backslash B(x, \delta).$$

Then

$$\begin{array}{rcl} -\Delta v^{-} & = & 0 \;\; \textit{in} \;\; \Omega, \\ v^{-}(x) & \leq & g(x) + |g(\xi) - g(x)| - \epsilon - kw(x) \\ & \leq & \left\{ \begin{array}{ll} g(x) + \epsilon - \epsilon - kw(x), & \textit{for} \;\; |x - \xi| < \delta \\ g(x) + 2M - \epsilon - 2M, & \textit{for} \;\; |x - \xi| \geq \delta \end{array} \right. \\ & < & g(x), \quad \forall x \in \partial\Omega. \end{array}$$

We prove in a similar way $g \leq v^{+}$, which proves the claim.
Claim: The solution is unique.
Proof. This is a direct consequence of the comparison principle, Corollary 4.3.2.

9.4 Distributions (Ch. 5)

9.4.1 Some useful inequalities

The following inequalities are useful when working with distributions, Fourier transform, L^p, and Sobolev spaces.

Hölder's inequality: $\forall p, p' \in [1,\infty], \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \forall (u,v) \in L^p(\Omega) \times L^{p'}(\Omega):$

$$\int_\Omega |u(x)v(x)|dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}, \tag{9.4.1}$$

Young's inequality: $\forall a, b \geq 0, \quad \forall p, p' \in [1,\infty], \quad \frac{1}{p} + \frac{1}{p'} = 1:$

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}, \tag{9.4.2}$$

Useful growth equiv. inequal: $\exists C_1, C_2 > 0, \ \forall \xi \in \mathbb{R}^N, \ \forall k \in \mathbb{N}, \ \forall \alpha \in \mathbb{N}_0^N, \ \forall s > 0:$

$$C_1(1 + |\xi|^{2k}) \leq \sum_{|\alpha| \leq k} |\xi^\alpha|^2 \leq C_2(1 + |\xi|^{2k}), \tag{9.4.3}$$

$$C_1(1 + |\xi|^{2s}) \leq (1 + |\xi|^2)^s \leq C_2(1 + |\xi|^{2s}). \tag{9.4.4}$$

9.4.2 More operations with distributions. Examples

Product of distributions

In general, the product $T_1 T_2$ of two distributions is not well-defined. For example, if $T_1 = T_2 = \dfrac{1}{\sqrt{x}}$ then T_1, $T_2 \in \mathcal{D}'$ but $T_1 T_2 \notin \mathcal{D}'$.

Even if $T \in \mathcal{D}'(\Omega)$ and $k \notin C^\infty(\Omega)$ then kT is not a distribution. For example, $T \in \mathcal{D}'(\mathbb{R})$, $\langle T, \varphi \rangle = \displaystyle\int_{\mathbb{R}} \frac{\varphi(x)}{\sqrt{|x|}} dx$, is a well-defined distribution. However, if $k(x) = \dfrac{1}{\sqrt{|x|}}$ then kT is not a distribution because the integral $\langle T, k\varphi \rangle = \displaystyle\int_{\mathbb{R}} \frac{\varphi(x)}{|x|} dx$, is not well-defined for all $\varphi \in \mathcal{D}(\mathbb{R})$, for example for φ such that $\varphi(0) \neq 0$. However, if $k \in C^\infty(\Omega)$ then $kT \in CD'(\Omega)$.

Change of variable

Let $\theta \in C^\infty(\mathbb{R}^N; \mathbb{R}^N)$ with inverse $\theta^{-1} \in C^\infty(\mathbb{R}^N; \mathbb{R}^N)$. If $T \in \mathcal{D}'(\mathbb{R}^N)$, we can define the distribution $T \circ \theta^{-1} \in \mathcal{D}'(\mathbb{R}^N)$ by

$$\langle T \circ \theta^{-1}, \varphi \rangle = \langle T, \varphi \circ \theta |\det[\nabla \theta]| \rangle. \tag{9.4.5}$$

The motivation for this definition is the following. Assume $T = f \in L^1(\mathbb{R}^N)$. Then

$$\langle T\circ\theta^{-1},\varphi\rangle = \langle f\circ\theta^{-1},\varphi\rangle = \int_{\mathbb{R}^N} f(\theta^{-1}(x))\varphi(x)dx = \int_{\mathbb{R}^N} f(x)(\varphi\circ\theta)(x)|det[\nabla\theta]|dx.$$

Example 9.4.1 *If $\theta(x) = Ax+b$ with $A = (a_{ij})_{i,j=1,N}$ a $N\times N$ invertible matrix, $b\in\mathbb{R}^N$ then $\theta^{-1}(x) = A^{-1}(x-b)$ and $T\circ\theta^{-1}$ is defined by*

$$\langle T\circ\theta^{-1},\varphi\rangle = \langle T,\varphi(Ax+b)|det(A)|\rangle.$$

In particular, if $\theta(x) = x+a$, $a\in\mathbb{R}^N$, then $\theta^{-1}(x) = x-a$, so $\langle T\circ(x-a),\varphi\rangle = \langle T,\varphi\circ(x+a)\rangle$. For $T=\delta_0$, we have

$$\langle \delta_0(x-a),\varphi\rangle = \langle\delta_0,\varphi(x+a)\rangle = \varphi(a).$$

We denote $\delta_a = \delta_0\circ(\cdot - a)$. Hence $\langle\delta_a,\varphi\rangle = \varphi(a)$.

Example 9.4.2 *Let $n\in\mathbb{N}$, $f\in\mathcal{D}'(\mathbb{R})$ and let us look for the solutions $T\in\mathcal{D}'(\mathbb{R})$ to*

$$x^nT = 0. \tag{9.4.6}$$

First we consider $n=1$. Then the solution is $T = C\delta_0$, with C constant. Indeed, clearly $T = C\delta_0$ is a solution. Conversely, for $\varphi\in\mathcal{D}$ set $\psi(x) = \dfrac{\varphi(x)-\varphi(0)\eta(x)}{x}$, where η is a $\{[-1,1],(-2,2)\}$ cut-off function. As $\psi(x) = \dfrac{\varphi(x)(1-\eta(x))}{x} + \eta(x)\displaystyle\int_0^1\varphi'(tx)dt$, we get $\psi\in\mathcal{D}$. Then

$$0 = \langle xT,\psi\rangle = \langle T,\varphi\rangle - \varphi(0)\langle T,\eta\rangle; \quad \textit{so} \quad \langle T,\varphi\rangle = \langle\langle T,\eta\rangle\delta_0,\varphi\rangle,$$

which proves the claim.

For the general case, we take $\psi(x) = \dfrac{1}{x^n}\left(\varphi(x) - \eta(x)\displaystyle\sum_{k=0}^{n-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right)$. Then $\psi\in\mathcal{D}(\mathbb{R})$, because $\psi(x) = \dfrac{\varphi(x)(1-\eta(x))}{x^n} + \dfrac{\eta(x)}{(n-1)!}\displaystyle\int_0^1\varphi^{(n)}(tx)(1-t)^{n-1}dt$. Then

$$\begin{aligned} 0 &= \langle x^nT,\psi\rangle = \langle T,x^n\psi\rangle \\ &= \langle T,\varphi\rangle - \sum_{k=0}^{n-1}\frac{\varphi^{(k)}(0)}{k!}\left\langle T,\eta x^k\right\rangle; \quad \textit{hence} \\ T &= \sum_{k=0}^{n-1} C_k\delta_0^{(k)}, \quad C_k \ \textit{constants.} \end{aligned}$$

Example 9.4.3 *Let $f \in \mathscr{D}'(\mathbb{R})$ and let us look for the solutions $T \in \mathscr{D}'(\mathbb{R})$ to*

$$xT = f. \tag{9.4.7}$$

Note that we have already seen the cases $f = 0$ and $f = 1$. If we know one particular solution S to (9.4.7), then all the solutions are given by $T = C\delta_0 + S$, C constant.

Now let us find an S. Inspired by (5.2.13), we define $\mathrm{pv}(x^{-1}f)$ by

$$\langle \mathrm{pv}(x^{-1}f), \varphi\rangle = \langle f, x^{-1}\varphi(1-\eta)\rangle + \langle f, x^{-1}(\varphi - \varphi(0))\eta\rangle, \tag{9.4.8}$$

where η is a $\{\{0\}, (-\epsilon, \epsilon)\}$ cut-off function, $\epsilon > 0$. It is easy to show that $T = \mathrm{pv}(x^{-1}f) \in \mathscr{D}'$ and it solves (9.4.7). So, $S = \mathrm{pv}(x^{-1}f)$ is a particular solution to (9.4.7), and all solutions to (9.4.7) are of the form $T = C\delta_0 + \mathrm{pv}(x^{-1}f)$.

Note that the solution does not depend on ϵ or η because if η_i, $i = 1, 2$, is a $\{\{0\}, (-\epsilon_i, \epsilon_i)\}$ cut-off function and $S_i = \mathrm{pv}(x^{-1}T)$ is defined by (9.4.8) with $\eta = \eta_i$, then $S_1 - S_2 = C\delta_0$.

Example 9.4.4 *Let $f \in \mathscr{D}'(\mathbb{R})$ and look for $T \in \mathscr{D}'$, solution to*

$$T' = f. \tag{9.4.9}$$

For $\varphi \in \mathscr{D}$ set $\psi(x) = \int_{-\infty}^{x} \varphi(t)dt - \int_{-\infty}^{x} \varphi_1(t)dt\langle 1, \varphi\rangle$, where $\varphi_1 \in \mathscr{D}$ with $\int_{\mathbb{R}} \varphi_1(t)dt = 1$. Then $\psi \in \mathscr{D}$. Assuming T exists implies

$$-\langle f, \psi\rangle = \langle T, \psi'\rangle = \langle T, \varphi\rangle - \langle T, \varphi_1\rangle\langle 1, \varphi\rangle.$$

Hence, if $C = \langle T, \varphi_1\rangle$ we get

$$\langle T, \varphi\rangle = \langle C, \varphi\rangle - \langle f, \psi\rangle. \tag{9.4.10}$$

Every T given by (9.4.9) solves (9.4.9), so (9.4.10) gives all the solutions to (9.4.9).

As an application, let us find the solutions to

$$xT' = 0, \quad x \in \mathbb{R}, \;\; T \in \mathscr{D}'(\mathbb{R}). \tag{9.4.11}$$

From Example 9.4.2 we have $T' = D\delta_0$, D constant. From (9.4.10) with $f = D\delta_0$ we get

$$\begin{aligned}\langle T, \varphi\rangle &= \langle C, \varphi\rangle - \langle D\delta_0, \psi\rangle = \langle C, \varphi\rangle - D\int_{-\infty}^{0} \varphi(t)dt + D\int_{-\infty}^{0} \varphi_1(t)dt\langle 1, \varphi\rangle \\ &= \langle C, \varphi\rangle + \langle DH, \varphi\rangle,\end{aligned}$$

for certain constants C, D, $D \neq 0$. So $T = C + DH$, where H is the Heaviside function.

9.4.3 Convergence of distributions. Distributions of finite order

Definition 3), 5.2.9, is consistent, thanks to the following theorem.

Theorem 9.4.5 *Let $T_n \in \mathcal{D}'(\Omega)$ such that $\lim\limits_{n\to\infty}\langle T_n,\varphi\rangle$ exists for all $\varphi \in \mathcal{D}(\Omega)$. Set $T : \mathcal{D}(\Omega) \mapsto \mathbb{R}$, $\langle T,\varphi\rangle = \lim\limits_{n\to\infty}\langle T_n,\varphi\rangle$. Then $T \in \mathcal{D}'(\Omega)$ and $\lim\limits_{n\to\infty} T_n = T$ in $\mathcal{D}'(\Omega)$.*

Proof. We refer the reader to [23] or [22] for a proof, which uses the uniform boundedness principle (Banach-Steinhaus theorem). For a proof by contradiction, the reader can see [43]. □

Any distribution $T \in \mathcal{D}'(\Omega)$ restricted in any compact K is of finite order, with the order eventually tending to infinity as K tends to Ω; see [22, 23]. The following theorem shows that the distributions with compact support are of finite order.

Theorem 9.4.6 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $T \in \mathcal{D}'(\Omega)$ with compact support. Then T is of finite order; see (5.2.6).*

Proof. Let $G \Subset \Omega$ be open with $\mathrm{supp}(T) \subset G$, and $\eta \in \mathcal{D}$ a $\{\mathrm{supp}(T), G\}$ cut-off function. Then

$$\langle T,\varphi\rangle = \langle T,\eta\varphi\rangle + \langle T,(1-\eta)\varphi\rangle = \langle T,\eta\varphi\rangle, \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Now, assume T is not of finite order. Then

$$\forall m \in \mathbb{N}, \quad \exists\varphi_m \in \mathcal{D}(\Omega), \quad |\langle T,\varphi_m\rangle| = |\langle T,\eta\varphi_m\rangle| > m\|\varphi_m\|_{C^m(\Omega)}.$$

Set $\psi_m = \dfrac{1}{m}\dfrac{\eta\varphi_m}{\|\varphi_m\|_{C^m(\Omega)}}$. Then $|\langle T,\psi_m\rangle| > 1$. But $\lim_{m\to\infty}\psi_m = 0$ in $\mathcal{D}(\Omega)$, because $\mathrm{supp}(\psi_m) \subset \overline{G}$ and for given $k \in \mathbb{N}$ we have $\|\psi_m\|_{C^k(\Omega)} \le \dfrac{C_k}{m}$. Passing to the limit in $|\langle T,\psi_m\rangle| > 1$ implies $0 \ge 1$, which is a contradiction and proves the theorem.

Remark 9.4.7 *Let $T \in \mathcal{D}'(\Omega)$ with $\mathrm{supp}(T)$ compact. Then T can be extended to a linear "continuous" map $\overline{T} : C^\infty(\Omega) \mapsto \mathbb{R}$ as follows.*

Let $G \Subset \Omega$ with $\mathrm{supp}(T) \subset G$, and $\eta \in \mathcal{D}$ be a $\{\mathrm{supp}(T), G\}$ cut-off function. Every $\varphi \in C^\infty(\Omega)$ is written as $\varphi = \varphi\eta + \varphi(1-\eta)$, with $\mathrm{supp}(\varphi(1-\eta)) \subset N(T)$, which implies $\langle T,\varphi\rangle = \langle T,\eta\varphi\rangle$. Then $\overline{T}$, the extension of T, is defined by

$$\langle \overline{T},\varphi\rangle = \langle T,\eta\varphi\rangle, \quad \forall\varphi \in C^\infty(\Omega).$$

Clearly $\langle \overline{T},\varphi\rangle = \langle T,\varphi\rangle$, for every $\varphi \in \mathcal{D}(\Omega)$. Note also that $\overline{T}$ does not depend on the choice of η because if η_i, $i = 1,2$, are two cut-off functions as above then $\langle T,(\eta_1-\eta_2)\varphi\rangle = 0$ because $\mathrm{supp}(\eta_1-\eta_2) \subset N(T)$. Finally, it is easy to check that $\overline{T}$ is continuous in the sense:

$$\lim_{n\to\infty}\langle \overline{T},\varphi_n\rangle = 0, \quad \forall\varphi_n \in C^\infty(\Omega) \text{ such that } \forall\alpha \in \mathbb{N}_0^N, \ \lim_{n\to\infty} D^\alpha\varphi_n = 0 \text{ in } C^0_{loc}(\Omega).$$

9.4.4 Convolution of distributions

When dealing with the convolution of distributions, we have to consider functions of two variables; let us say $\varphi = \varphi(x, y) \in \mathcal{D}(\mathbb{R}^N \times \mathbb{R}^N)$. In order to identify which variable in the distribution $T \in \mathcal{D}'$ is active, we will write $T(x)$, or $T(y)$. For example, $\langle T(x), \varphi(x, y)\rangle := \langle T, \eta\rangle$, where $\eta(x) = \varphi(x, y)$, y fixed. Similarly, $\langle T(y), \varphi(x, y)\rangle := \langle T, \zeta\rangle$, where $\zeta(y) = \varphi(x, y)$, x fixed.

The following two auxiliary results show that the derivative and the integration commute with the distribution pairing. They are needed in order to prove that the convolution of two distributions is well-defined, as well as the density of $\mathcal{D}$ in $\mathcal{D}'$.

Lemma 9.4.8 (Derivative and distribution paring) *Let $T \in \mathcal{D}'(\mathbb{R}^N)$, $\varphi \in C^\infty(U \times \mathbb{R}^N)$, with $U \subset \mathbb{R}^N$ open.*

a) If φ satisfies: $\forall x \in U$, $\exists r > 0$ and $K \subset \mathbb{R}^N$ compact such that

$$i) \quad \forall \tilde{x} \in B(x, r), \quad \operatorname{supp}(\varphi(\tilde{x}, \cdot)) \subset K, \quad \textit{and}$$
$$ii) \quad \forall \alpha, \beta \in \mathbb{N}_0^N, \quad \|D_x^\alpha D_y^\beta \varphi\|_{C^0(B(x,r)\times\mathbb{R}^N)} < \infty,$$

$$\textit{then} \quad \begin{cases} \langle T(y), \varphi(\cdot, y)\rangle \in C^\infty(U), \\ D^\alpha\langle T(y), \varphi(x, y)\rangle = \langle T(y), D_x^\alpha \varphi(x, y)\rangle, \quad \forall x \in U, \ \forall \alpha \in \mathbb{N}_0^N. \end{cases} \tag{9.4.12}$$

b) Furthermore, if T has compact support and $\varphi(x, y) := \varphi(x+y)$, where $\varphi \in \mathcal{D}(\mathbb{R}^N)$, then (9.4.12) holds, $\langle T(y), \varphi(\cdot + y)\rangle \in \mathcal{D}$ and $\operatorname{supp}(\langle T(y), \varphi(\cdot + y)\rangle) \subset \operatorname{supp}(\varphi) - \operatorname{supp}(T)$.

Proof. Let us prove (9.4.12) for $|\alpha| = 1$, for example $\alpha = (1, 0, \dots, 0)$. For $x \in U$, let r and K be as in i) and ii) above. Let $e = (1, 0, \dots, 0)$, $0 \neq h \in \mathbb{R}$, and consider

$$\begin{aligned} z(h) &= \frac{\langle T(y), \varphi(x + he, y)\rangle - \langle T(y), \varphi(x, y)\rangle}{h} - \langle T(y), D_x^\alpha \varphi(x, y)\rangle \\ &= \langle T(y), \epsilon_h(y)\rangle, \quad \textit{with} \\ \epsilon_h(y) &:= \frac{1}{h}\left(\varphi(x + he, y) - \varphi(x, y) - hD_x^\alpha \varphi(x, y)\right). \end{aligned}$$

Note that $\operatorname{supp}(\epsilon_h) \subset K$ for $|h| < r$. Next, for every $\beta \in \mathbb{N}_0^N$ we have

$$\begin{aligned} |D^\beta \epsilon_h(y)| &= \left|\frac{1}{h}\left(D_y^\beta \varphi(x + he, y) - D_y^\beta \varphi(x, y) - hD_x^\alpha D_y^\beta \varphi(x, y)\right)\right| \\ &= \left|\frac{1}{2}hD_x^{2\alpha} D_y^\beta \varphi(x + the, y)\right| \leq \frac{1}{2}h\|D_x^{2\alpha} D_y^\beta \varphi\|_{C^0(B(x,r)\times\mathbb{R}^N)} \qquad (\textit{here } t \in (0, 1)) \\ &\xrightarrow{h\to 0} 0, \end{aligned}$$

which proves $\lim_{h\to 0} \epsilon_h = 0$ in $\mathcal{D}$. Hence, $\lim_{h\to 0} z(h) = 0$, which proves (9.4.12) for $|\alpha| = 1$. One can prove (9.4.12) for arbitrary $|\alpha| \neq 1$ by induction. This completes the proof of Claim a).

For Claim b), we note that (9.4.12) holds because $\varphi(x, y)$ satisfies the condition of i) and ii). It remains to show that $\langle T(y), \varphi(\cdot + y)\rangle$ has compact support. Let $G \subset \mathbb{R}^N$ be open with $\mathrm{supp}(T) \subset G$ and η a $\{\mathrm{supp}(T), G\}$ cut-off function. Then

$$\psi(y) := \langle T(y), \varphi(x + y)\rangle \quad = \quad \langle T(y), \eta(y)\varphi(x + y)\rangle.$$

Clearly, if x is such that $\eta(\cdot)\varphi(x + \cdot) = 0$ then $\psi(x) = 0$. This implies $\mathrm{supp}(\psi) \subset \{x,\ \eta(\cdot)\varphi(x + \cdot) \neq 0\} \subset \{x \in \mathrm{supp}(\varphi) - \mathrm{supp}(\eta)\}$, where in general $A - B = \cup\{p - q,\ p \in A,\ q \in B\}$ for any $A, B \subset \mathbb{R}^N$. As $\mathrm{supp}(\eta) \subset G$, and G is an arbitrary open set including $\mathrm{supp}(T)$, we obtain that $x \in \mathrm{supp}(\varphi) - \mathrm{supp}(T)$. So $\mathrm{supp}(\langle T(y), \varphi(\cdot + y)\rangle) \subset \mathrm{supp}(\varphi) - \mathrm{supp}(T)$. □

The following two results are corollaries of this lemma.

Lemma 9.4.9 (Integration and distributing paring) *Let $T \in \mathcal{D}'$, $\phi \in \mathcal{D}(\mathbb{R}^N \times \mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} \langle T(y), \phi(x, y)\rangle dx = \left\langle T(y), \int_{\mathbb{R}^N} \phi(x, y)dx \right\rangle. \tag{9.4.13}$$

Proof. Note that from Fubini's theorem 9.1.3, we have

$$\int_{\mathbb{R}^N} \phi(x, y)dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \phi(z_1, \dots, \dots, z_N, y)dz_N \dots dz_1. \tag{9.4.14}$$

Set $\varphi(x, y) = \displaystyle\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_N} \phi(z_1, \dots, z_N, y)dz_N \cdots dz_1$. The function φ satisfies the conditions of Lemma 9.4.8 with $U = \mathbb{R}^N$ and $K = \overline{\{y \in \mathbb{R}^N, \phi(\cdot, y) \neq 0\}}$ regardless of x, which implies that $\langle T(y), \varphi(\cdot, y)\rangle$ is a $C^\infty(\mathbb{R}^N)$ function and

$$\langle T(y), \phi(x, y)\rangle = \langle T(y), \partial_{x_1} \cdots \partial_{x_N} \varphi(x, y)\rangle = \partial_{x_1} \cdots \partial_{x_N} \langle T(y), \varphi(x, y)\rangle.$$

Integrating this equality in $\mathbb{R}^N$ gives (9.4.13).

Proposition 9.4.10 *$T * S$ as defined by (5.3.1) is an element of $\mathcal{D}'$.*

Proof. Indeed, let $\varphi \in \mathcal{D}$. From Lemma 9.4.8, $\langle T(y), \varphi(\cdot + y)\rangle \in \mathcal{D}$ and $\mathrm{supp}(\langle T(y), \varphi(\cdot + y)\rangle) \subset \mathrm{supp}(\varphi) - \mathrm{supp}(T)$, so $\langle T * S, \varphi\rangle$ is well-defined for every φ.

Next, we prove that if (φ_n) is a sequence in $\mathcal{D}$ with $\lim_{n\to\infty} \varphi_n = 0$ in $\mathcal{D}$, then $\lim_{n\to\infty}\langle T * S, \varphi_n\rangle = 0$. For this, we note that if $\psi_n(x) = \langle S(y), \varphi_n(x + y)\rangle$), clearly we have $\psi_n \in \mathcal{D}$ with $\mathrm{supp}(\psi_n) \subset \mathrm{supp}(\varphi_n) - \mathrm{supp}(T) \subset K - \mathrm{supp}(T)$, where K is a compact including the support of all φ_n. Furthermore, for $\alpha \in \mathbb{N}_0^N$ we have $D^\alpha\psi_n(x) = \langle S(y), D^\alpha\varphi_n(x + y)\rangle$, and as S is of finite order (see Theorem 5.2.15), say m, we have

$$|D^\alpha\psi_n(x)| \leq |\langle S(y), D^\alpha\varphi_n(x + y)\rangle| \leq C\|\varphi_n\|_{C^{m+|\alpha|}(\mathbb{R}^N)},$$

which proves $\lim_{n\to\infty} \psi_n = 0$ in $\mathcal{D}$. Hence $\lim_{n\to\infty}\langle T * S, \varphi_n\rangle = \lim_{n\to\infty}\langle T, \psi_n\rangle = 0$.

9.4.5 Tempered distributions and Fourier transform

Proposition 9.4.11 *We have*

i) $\{x^\alpha D^\beta u,\ u \in \mathcal{S}\} \subset \mathcal{S}$, *for all* $\alpha, \beta \in \mathbb{N}_0^N$.

ii) $\mathcal{D} \subset \mathcal{S}$, *with dense inclusion, i.e. for every* $u \in \mathcal{S}$ *there exists* (u_n) *in* $\mathcal{D}$ *such that* $\lim_{n\to\infty} u_n = u$ *in* $\mathcal{S}$.

iii) $\mathcal{S} \subset L^p$, $p \in [1, \infty]$, *with dense inclusion for* $p \in [1, \infty)$, *i.e. for every* $u \in L^p$ *there exists* (u_n) *in* $\mathcal{S}$ *such that* $\lim_{n\to\infty} u_n = u$ *in* L^p.

iv) $L^p \subset \mathcal{S}'$, $p \in [1, \infty]$, *with dense inclusion, i.e. for every* $u \in \mathcal{S}'$ *there exists* (u_n) *in* L^p *such that* $\lim_{n\to\infty} u_n = u$ *in* $\mathcal{S}'$.

v) $\mathcal{S}' \subset \mathcal{D}'$ *with dense inclusion, i.e. for every* $u \in \mathcal{D}'$ *there exists* (u_n) *in* $\mathcal{D}'$ *such that* $\lim_{n\to\infty} u_n = u$ *in* $\mathcal{D}'$.

Proof of i). Set $v = x^\alpha D^\beta u$. For arbitrary $a, b \in \mathbb{N}_0^n$, we have

$$|x^a D^b v| \leq C(b)|x^a| \sum_{|c|\leq|b|} D^c x^\alpha D^{\beta+b-c} u| \leq C(\alpha, \beta, a, b) s_m(u),$$

with $m = |\alpha| + |a| + |\beta| + |b|$, which proves i).

Proof of ii). Clearly $\mathcal{D} \subset \mathcal{S}$. Let us prove that the inclusion is dense. Let $u \in \mathcal{S}$ and $\eta \in \mathcal{D}$ be a $\{\overline{B}(0,1), B(0,2)\}$ cut-off function. If $u_n(x) = \eta(x/n)u(x)$ then $u_n \in \mathcal{D}$. Furthermore, for $\alpha, \beta \in \mathbb{N}_0^N$ we have

$$\begin{aligned}
|x^\alpha D^\beta(u_n - u)| &= |x^\alpha D^\beta((1 - \eta(x/n))u)| \\
&\leq C(\beta)|x^\alpha| \left(|1 - \eta(x/n)||D^\beta u| + \sum_{0<\gamma\leq\beta} |D^\gamma(\eta(x/n))D^{\beta-\gamma}u| \right) \\
&\leq C(\beta)|x|^{|\alpha|} \Big(\|\eta\|_{C^1(\mathbb{R}^N)} \frac{|x|}{n} |D^\beta u| + \\
&\qquad \|\eta\|_{C^{|\beta|}(\mathbb{R}^N)} \frac{1}{n} \sum_{0<\gamma\leq\beta} |x|^{|\gamma|} |D^{\beta-\gamma}u| \Big) \\
&\leq \frac{C(\beta)}{n} \|\eta\|_{C^{1+|\beta|}(\mathbb{R}^N)} s_m(u) \\
&\xrightarrow{n\to\infty} 0,
\end{aligned}$$

with $m = 1 + |\alpha| + |\beta|$, which proves $\lim\limits_{n\to\infty} u_n = u$ in $\mathcal{S}$.

Proof of iii). Note that $\zeta(x) = (1 + |x|^2)^{-N} \in L^p$, $p \in [1, \infty]$, and

$$|u(x)| = |\zeta(x)(1 + |x|^2)^N u| \leq C|\xi(x)| s_{2N}(u),$$

which proves the inclusion $\mathcal{S} \subset L^p$, $p \in [1, \infty]$. The density of the inclusion for $p \in [1, \infty)$ follows from ii) and Theorem 1.3.5.

Proof of iv). We have $L^p \subset \mathcal{S}'$ in the sense that if $u \in L^p$ then T_u defined by $\langle T_u, \varphi \rangle = \int_{\mathbb{R}^N} u\varphi dx$, $\varphi \in \mathcal{S}$, defines an element of $\mathcal{S}'$. It is easy to verify that, indeed, $T_u \in \mathcal{S}'$. For the density of $L^p \subset \mathcal{S}'$, one can prove that $\mathcal{S} \subset \mathcal{S}'$, with dense inclusion in the sense that for every $T \in \mathcal{S}'$ there exists (T_n) in $\mathcal{S}$ such that $\lim_{n\to\infty} T_n = T$ in $\mathcal{S}'$. The proof of this result is technical and we will omit the details. It follows the same steps as the proof of the dense embedding $\mathcal{D} \subset \mathcal{D}'$; see Theorem 5.3.3. Namely, take $T_n(x) = \langle T(y), \rho_n(x-y) \rangle$, where (ρ_n) is a regularizing sequence with $\rho_n(x) = \rho_n(-x)$. Next, like in Lemma 9.4.8 one can show that $T_n \in \mathcal{S}$. Furthermore, like in Lemma 9.4.9 one can prove that $\langle T_n, \varphi \rangle = \langle T, \rho_n * \varphi \rangle$, which shows that $T_n \to T$ in $\mathcal{S}'$.

Proof of v). We have $\mathcal{S}' \subset \mathcal{D}'$ in the sense that if $T \in \mathcal{S}'$, then T restricted in $\mathcal{D}$ is an element of $\mathcal{D}'$. For the density of the inclusion, let η_n be a $\{\overline{B}(0,n), B(0,2n)\}$ cut-off function, $n \in \mathbb{N}$. For $T \in \mathcal{D}'$ consider $T_n = \eta_n T$. Then $T_n \in \mathcal{S}'$ and $\lim_{n\to\infty} T_n = T$ in $\mathcal{D}'$.

Proposition 9.4.12 *Let $T \in \mathcal{S}'$. Then T satisfies*

$$\exists C > 0,\ \exists m \in \mathbb{N}_0, \quad \forall \varphi \in \mathcal{S}, \quad |\langle T, \varphi \rangle| \leq C s_m(\varphi). \tag{9.4.15}$$

For every $T \in \mathcal{S}'$ satisfying (9.4.15) we say "T has finite order", and m is the order of T.

Proof. The proof is similar to the proof of Theorem 5.2.15 (Theorem 9.4.6). Indeed, assume (9.4.15) does not hold. Then

$$\forall n \in \mathbb{N}, \quad \exists \varphi_n \in \mathcal{S}, \quad |\langle T, \varphi_n \rangle| > n s_n(\varphi_n).$$

Set $\psi_n = \dfrac{1}{n}\dfrac{\varphi_n}{s_n(\varphi_n)}$. Then $|\langle T, \psi_n \rangle| > 1$ and $\lim_{n\to\infty} \psi_n = 0$ in $\mathcal{S}$ because $s_m(\psi_n) \leq n^{-1}$ for $n > m$. Passing to the limit in $|\langle T, \psi_n \rangle| > 1$ implies $0 \geq 1$, which is a contradiction and proves the proposition.

Theorem 9.4.13 *Fourier transform $\mathcal{F}$ is a linear continuous bijection from $\mathcal{S}$ to itself. Its inverse, denoted by $\mathcal{F}^{-1}$, is continuous from $\mathcal{S}$ to itself, and is given by*

$$\mathcal{F}^{-1}[U](x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i(x\cdot\xi)} U(\xi) d\xi, \quad \forall U \in \mathcal{S}. \tag{9.4.16}$$

Furthermore, similar to (5.4.4), (5.4.5), for all $u \in \mathcal{S}'$ and α, $\beta \in \mathbb{N}_0^N$ we have

$$D^\alpha u(x) = \mathcal{F}^{-1}[(i\xi)^\alpha \hat{u}](x), \tag{9.4.17}$$

$$(-ix)^\beta u(x) = \mathcal{F}^{-1}[D^\beta \hat{u}](x). \tag{9.4.18}$$

Proof.
i) Let us prove first $\mathcal{F}[\mathcal{S}] \subset \mathcal{S}$ and the continuity of $\mathcal{F}$. For $u \in \mathcal{S}$ let $\hat{u} = \mathcal{F}[u]$. Note that $\hat{u} \in C^\infty$. From (5.4.6), we get

$$\begin{aligned}
|\xi^\alpha D^\beta \hat{u}(\xi)| &\leq C\|D^\alpha(x^\beta u)\|_{L^1(\mathbb{R}^N)} \\
&\leq C(\alpha) \sum_{|\gamma|\leq|\alpha|} \|D^\gamma x^\beta D^{\alpha-\gamma} u\|_{L^1(\mathbb{R}^N)} \\
&\leq C(\alpha) \sum_{|\gamma|\leq|\alpha|} \left\| \frac{1}{(1+|x|^2)^N} \right\|_{L^1(\mathbb{R}^N)} \left\|(1+|x|^2)^N D^\gamma x^\beta D^{\alpha-\gamma} u\right\|_{L^\infty(\mathbb{R}^N)} \\
&\leq C(\alpha, N) s_m(u), \;\; m = |\alpha| + |\beta| + 2N, \qquad (9.4.19)
\end{aligned}$$

which proves $\hat{u} \in \mathcal{S}$ and the continuity of $\mathcal{F}$.
ii) Now let us prove (9.4.16).[7] Let $u \in \mathcal{S}$, $\hat{u} = \mathcal{F}[u]$ and for $z \in \mathbb{R}^N$ set

$$v(z) = \mathcal{F}^{-1}[\hat{u}](z) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i(\xi\cdot z)} \hat{u}(\xi) d\xi = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i(z-x)\cdot\xi} u(x) dx d\xi.$$

We want to prove $v(z) = u(z)$. For this, we want to use Fubini's theorem in the previous formula, which is not allowed because $e^{i(z-x)\cdot\xi} \notin L^1$. However, we can proceed as follows. For $\epsilon > 0$ set

$$\begin{aligned}
v_\epsilon(z) &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i(z\cdot\xi)} e^{-\epsilon|\xi|^2} \hat{u}(\xi) d\xi \\
&= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(x) \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i(x-z)\cdot\xi} e^{-\epsilon|\xi|^2} dx d\xi.
\end{aligned}$$

Then from Lebesgue's DCT, we have $\lim\limits_{\epsilon\to 0} v_\epsilon(z) = v(z)$. Now, for v_ϵ we can use Fubini's theorem, which together with (5.4.10) gives

$$\begin{aligned}
v_\epsilon(z) &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(x) \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i(x-z)\cdot\xi} e^{-\epsilon|\xi|^2} d\xi dx \\
&= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(x) \mathcal{F}[e^{-\epsilon\xi^2}](x-z) dx \\
&= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(z+x) \mathcal{F}[e^{-\epsilon|\xi|^2}](x) dx \\
&= \frac{1}{2^N \pi^{N/2} \epsilon^{N/2}} \int_{\mathbb{R}^N} u(z+x) e^{-\frac{1}{4\epsilon}x^2} dx \qquad \textit{(using (5.4.10))} \\
&= \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} u(z+2x\sqrt{\epsilon}) e^{-x^2} dx \\
&\xrightarrow{\epsilon\to 0} u(z) \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} e^{-x^2} dx \qquad \textit{(using (5.4.7))}
\end{aligned}$$

[7] See Corollary 5.4.15 for another proof.

$$= \quad u(z).$$

Hence $\mathscr{F}^{-1}[\hat{u}] = u$. It follows $\mathscr{F}[\mathscr{S}] = \mathscr{S}$, because for every $u \in \mathscr{S}$ we have

$$u(x) = \mathscr{F}^{-1}[\mathscr{F}[u](\xi)](x) = \mathscr{F}^{-1}[\hat{u}(\xi)](x) = \mathscr{F}[\hat{u}(-\xi)](x)$$

and $\hat{u}(-\xi) \in \mathscr{S}$, which proves that $\mathscr{F}$ is invertible from $\mathscr{S}$ to itself, and its inverse $\mathscr{F}^{-1}$ is given by (9.4.16).
iii) Finally, (9.4.17) and (9.4.18) follow by simple calculations. □

Theorem 5.4.9 shows that $\mathscr{F}$ is an isometry in L^2. The following two results deal with the action of $\mathscr{F}$ in L^p spaces.

Lemma 9.4.14 (Riemann-Lebesgue lemma) *Let $u \in L^1$. Then*

$$\|\hat{u}\|_{L^\infty} \leq (2\pi)^{-N/2}\|u\|_{L^1}, \quad \hat{u} \in C^0 \cap L^\infty, \quad \lim_{|\xi|\to\infty} \hat{u}(\xi) = 0. \tag{9.4.20}$$

Proof. The L^∞ bound follows from

$$|\hat{u}(\xi)| \leq (2\pi)^{-N/2} \int_{\mathbb{R}^N} |e^{-i(\xi\cdot x)}u(x)|dx \leq (2\pi)^{-N/2}\|u\|_{L^1(\mathbb{R}^N)}.$$

The continuity of $\hat{u}$ follows from the Lebesgue DCT, because

$$\lim_{\eta\to\xi}(e^{-i(\xi\cdot x)} - e^{-i(\eta\cdot x)})u(x) = 0, \ \forall x \in \mathbb{R}^N, \quad |e^{-i(\xi\cdot x)}u(x)| \leq |u(x)| \in L^1.$$

For the limit at infinity, assume first that $u \in \mathscr{D}$. Note that $D^\alpha u \in \mathscr{D}$ for all $\alpha \in \mathbb{N}_0^N$. Therefore, from (5.4.5) we have $\mathscr{F}[\Delta u] = -\xi^2\hat{u}$. Then from (9.4.20) we get

$$|\hat{u}(\xi)| \leq \frac{\|\mathscr{F}[\Delta u]\|_{L^\infty}}{\xi^2} \leq \frac{\|\Delta u\|_{L^1}}{(2\pi)^{N/2}} \cdot \frac{1}{\xi^2}, \quad |\xi| \neq 0, \tag{9.4.21}$$

which proves the limit in (9.4.20) for $u \in \mathscr{D}$.

Now let $u \in L^1$. From the density of $\mathscr{D}$ in L^1, for $\epsilon > 0$ there exists $\varphi \in \mathscr{D}$ such that

$$\|u - \varphi\|_{L^1(\mathbb{R}^N)} < (2\pi)^{N/2}\frac{\epsilon}{2}.$$

From (9.4.21) and the bound (9.4.20), it follows

$$|\hat{u}(\xi)| \leq |(\hat{u} - \hat{\varphi})(\xi)| + |\hat{\varphi}(\xi)| \leq (2\pi)^{-N/2}\|u - \varphi\|_{L^1} + \frac{\|\Delta\varphi\|_{L^1(\mathbb{R}^N)}}{(2\pi)^{N/2}}\frac{1}{\xi^2} < \epsilon,$$

for $|\xi|^2 > \dfrac{2}{\epsilon}\dfrac{\|\Delta\varphi\|_{L^1}}{(2\pi)^{N/2}}$, which proves the limit in (9.4.20). □

The action of $\mathscr{F}$ in L^p spaces is more complex. For $p \in [1, 2]$ we have this result.

Theorem 9.4.15 *For $p \in [1,2]$, $\mathcal{F}$ maps continuously L^p in L^q, where $1/p + 1/q = 1$.*

Proof. For $p = 1$ and $q = \infty$ the theorem is proven in Lemma 9.4.14, and for $p = 2$ and $q = 2$ in Theorem 5.4.9. For $p \in (1,2)$ and $q \in (2,\infty)$, the theorem follows from Riesz-Thorin interpolation theorem 9.4.16 with $p_0 = 1$, $q_0 = \infty$, $p_1 = 2$, $q_1 = 2$, $1/p = (1-\theta)/p_0 + \theta/p_1 = (2-\theta)/2$, $1/q = (1-\theta)/q_0 + \theta/q_1 = \theta/2$, and $T = \mathcal{F}$. □

Note that for $u \in L^p$, $p \in (2,\infty)$, in general $\mathcal{F}[u]$ is not a function; see [49].

Theorem 9.4.16 (Riesz-Thorin interpolation theorem) *Let $1 \le p_0, q_0, p_1, q_1 \le \infty$ and $T : L^{p_0} + L^{p_1} \mapsto L^{q_0} + L^{q_1}$ be a bounded linear operator from L^{p_0} into L^{q_0} and L^{p_1} into L^{q_1}. For $\theta \in (0,1)$, let p_θ and q_θ be defined by $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$, $\dfrac{1}{q_\theta} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}$. Then T is linear and bounded from L^{p_θ} into L^{q_θ} and satisfies*

$$\|T\|_{\mathcal{L}(L^{p_\theta};L^{q_\theta})} \le \|T\|^{1-\theta}_{\mathcal{L}(L^{p_0};L^{q_0})} \|T\|^{\theta}_{\mathcal{L}(L^{p_1};L^{q_1})}. \tag{9.4.22}$$

See [49] for the proof.

9.4.6 Tempered distributions and convolution

In this section, we will consider the convolution of tempered distributions.

Theorem 9.4.17 *The convolution operator $*$ is continuous from $\mathcal{S} \times \mathcal{S}$ onto $\mathcal{S}$. Moreover, for $u, v \in \mathcal{S}$ we have*

$$\begin{aligned} u * v &= v * u \in \mathcal{S}, && (9.4.23)\\ \mathcal{F}[u * v] &= (2\pi)^{N/2}\mathcal{F}[u]\mathcal{F}[v], && (9.4.24)\\ \mathcal{F}^{-1}[uv] &= (2\pi)^{-N/2}\mathcal{F}^{-1}[u] * \mathcal{F}^{-1}[v]. && (9.4.25) \end{aligned}$$

Proof. From Theorem 5.2.5 we have $u * v \in L^1 \cap C^\infty$, and for $\alpha, \beta \in \mathbb{N}_0^N$ we get

$$\begin{aligned} |x^\alpha D^\beta (u * v)(x)| &= \left| \int_{\mathbb{R}^N} x^\alpha D^\beta u(x-y) v(y) dy \right| \\ &= \left| \int_{\mathbb{R}^N} (x-y+y)^\alpha D^\beta u(x-y) v(y) dy \right| \\ &\le C(\alpha) \sum_{|\sigma|,|\gamma| \le |\alpha|} \int_{\mathbb{R}^N} |(x-y)^\gamma D^\beta u(x-y)||y^\sigma v(y)| dy \\ \textit{(as in Thm. 5.4.8)} \quad &\le C(\alpha, N) s_{|\alpha|+|\beta|}(u) s_{|\alpha|+2N}(v), \end{aligned}$$

which proves (9.4.23) and the continuity of $*$ in $\mathcal{S} \times \mathcal{S}$.

For (9.4.24), we use Fubini's theorem as follows:

$$\mathcal{F}[u * v](\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i(\xi \cdot x)} \int_{\mathbb{R}^N} u(x-y) v(y) dy dx$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} v(y) \int_{\mathbb{R}^N} e^{-i(\xi\cdot x)} u(x-y)\,dxdy \\
&= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i(\xi\cdot y)} v(y) \int_{\mathbb{R}^N} e^{-i(\xi\cdot(x-y))} u(x-y)\,dxdy \\
&= (2\pi)^{N/2} \mathcal{F}[u](\xi)\mathcal{F}[v](\xi),
\end{aligned}
$$

which proves (9.4.24). Applying $\mathcal{F}^{-1}$ to (9.4.24) gives

$$\mathcal{F}^{-1}[\hat{u}\hat{v}] = (2\pi)^{-N/2} u * v,$$

which together with the fact $\mathcal{F}[\mathcal{S}] = \mathcal{S}$ proves (9.4.25). □

For the convolution in $\mathcal{S}'$, we avoid the technical details and will only give the formal definition of the convolution and some properties. For the details of the proofs, the reader can consult [12, 18, 22, 23].

Definition 9.4.18 *Let $T \in \mathcal{S}'$ and $S \in \mathcal{E}'$, where $\mathcal{E}'$ is the set of linear maps from $C^\infty(\mathbb{R}^N)$ to $\mathbb{C}$ continuous for the convergence in $C^\infty(\mathbb{R}^N)$.*[8] *We define the convolution $T * S : \mathcal{S} \mapsto \mathbb{C}$ by*

$$\langle T * S, \varphi\rangle = \langle T(x), \langle S(y), \varphi(x+y)\rangle\rangle, \quad \forall \varphi \in \mathcal{S}. \tag{9.4.26}$$

Theorem 9.4.19 *Let $T \in \mathcal{S}'$ and $S \in \mathcal{E}'$. Then*

$$
\begin{aligned}
T * S &= S * T \in \mathcal{S}', && (9.4.27)\\
D^\alpha[T * S] &= D^\alpha T * S = T * D^\alpha S, && (9.4.28)\\
\mathcal{F}[T * S] = \mathcal{F}[S * T] &= (2\pi)^{N/2}\mathcal{F}[T]\mathcal{F}[S]. && (9.4.29)
\end{aligned}
$$

[8] A sequence (φ_n) in $C^\infty(\mathbb{R}^N)$ converges to 0 in $C^\infty(\mathbb{R}^N)$ if for every compact $K \subset \mathbb{R}^N$ and for every $\alpha \in \mathbb{N}_0^N$, $\lim_{n\to\infty} \|D^\alpha \varphi_n\|_{C^0(K)} = 0$.

9.5 Sobolev spaces (Ch. 6)

9.5.1 Continuous and compact embeddings

Theorem 9.5.1 *Assume $s-\frac{N}{2}\in(m,m+1]$, $m\in\mathbb{N}_0$ and let $\sigma\in\big(0,s-\frac{N}{2}-m\big)$. Then*

$$H^s \hookrightarrow C_b^{m,\sigma}\cap\{u,\ \lim_{|x|\to\infty}D^\alpha u=0,\ \forall\alpha\in\mathbb{N}_0^N,\ |\alpha|\le m\}. \tag{9.5.1}$$

In particular, there exists $C>0$ such that for all $\alpha\in\mathbb{N}_0^N$ with $|\alpha|=m$ and $x,y\in\mathbb{R}^N$ we have

$$|D^\alpha u(x)-D^\alpha u(y)|\le C\|D^\alpha u\|_{H^{s-m}}|x-y|^\sigma. \tag{9.5.2}$$

Proof. Let us consider first the case $m=0$. From Theorem 6.3.3 we have $H^s\hookrightarrow C_b^0$. To complete the proof in this case, it is enough to prove (9.5.2). It is easy to point out that for σ as in theorem and arbitrary $x,y,\xi,\eta\in\mathbb{R}^N$, we have[9]

$$|e^{i(\xi\cdot x)}-e^{i(\xi\cdot y)}|=\left|\int_0^1\frac{d}{dt}e^{i\xi\cdot(y+t(x-y))}dt\right|\le 2|\xi|^\sigma|x-y|^\sigma. \tag{9.5.3}$$

It follows that

$$\begin{aligned}
|u(x)-u(y)| &\le \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}|e^{i(x\cdot\xi)}-e^{i(y\cdot\xi)}||\hat u|d\xi\\
&\le \frac{2}{(2\pi)^{N/2}}|x-y|^\sigma\int_{\mathbb{R}^N}|\xi|^\sigma|\hat u|d\xi\\
&\le \frac{2}{(2\pi)^{N/2}}|x-y|^\sigma\int_{\mathbb{R}^N}|\xi|^\sigma\langle\xi\rangle^{-s}\langle\xi\rangle^s|\hat u|d\xi\\
&\le \frac{2}{(2\pi)^{N/2}}|x-y|^\sigma\||\xi|^\sigma\langle\xi\rangle^{-s}\|_{L^2}\|\langle\xi\rangle^s\hat u\|_{L^2}\\
&= C|x-y|^\sigma\|u\|_{H^s},
\end{aligned} \tag{9.5.4}$$

with $C=\frac{2}{(2\pi)^{N/2}}\||\xi|^\sigma\langle\xi\rangle^{-s}\|_{L^2}$. Note that we have (as in Theorem 6.3.3)

$$\||\xi|^\sigma\langle\xi\rangle^{-s}\|_{L^2}^2\sim\left(1+\int_1^\infty r^{N+2(\sigma-s)-1}dr\right)<\infty,$$

which proves the theorem in the case $m=0$.

The general case $m>0$ follows from the case $m=0$ by applying it to $D^\alpha u$, with $|\alpha|\le m$, and noting that $D^\alpha u\in H^{s-|\alpha|}$ (see Example 6.2.4) and $(s-|\alpha|)-\frac{N}{2}>0$.

Theorem 9.5.2 *Let $k\in\mathbb{N}$ and assume Ω is an open bounded set satisfying the $H^{k+1}(\Omega)$-extension property. Then $H^{k+1}(\Omega)\overset{c}{\hookrightarrow}H^k(\Omega)$.*

[9] Actually, this inequality holds for every $\sigma\in(0,1)$.

Proof. Let (u_n) be a bounded sequence in $H^{k+1}(\Omega)$, i.e. $\|u_n\|_{H^{k+1}(\Omega)} \leq C_1$ for all n. From Theorem 6.3.7, there exists a sequence $U_n = Eu_n \in H^{k+1}(\mathbb{R}^N)$, with $\|U_n\|_{H^{k+1}(\mathbb{R}^N)} \leq C_2 := C_1\|E\|$ and E an H^{k+1} extension operator. Without loss of generality we may assume that $\operatorname{supp}(U_n) \subset K$, K compact, because we can multiply (U_n) with $\eta(Rx)$, η a $\{\overline{B}(0,1), B(0,2)\}$ cut-off function, and $R >> 1$.

The sequence $(\hat{U}_n)$, $\hat{U}_n = \mathcal{F}[U_n]$, is uniformly bounded and uniformly Lipschitz in $\mathbb{R}^N$. Indeed, as U_n are with compact support we have $U_n \in L^1(\mathbb{R}^N)$, and from Lemma 9.4.14 obtain $\hat{U}_n \in C^0(\overline{\mathbb{R}}^N)$ and

$$|\hat{U}_n(\xi)| \leq \frac{1}{(2\pi)^{N/2}} \int_K |U_n| dx \leq \frac{1}{(2\pi)^{N/2}} C_2 |K|^{1/2} =: C_3. \tag{9.5.5}$$

Furthermore, $\hat{U}_n$ satisfies

$$\begin{aligned} |\hat{U}_n(\xi) - \hat{U}_n(\eta)| &= \frac{1}{(2\pi)^{N/2}} \left| \int_K (e^{-i(\xi\cdot x)} - e^{-i(\eta\cdot x)}) U_n(x) dx \right| \\ &\leq \frac{|\xi-\eta|}{(2\pi)^{N/2}} \int_K |x||e^{-i(\eta+\theta(\xi-\eta))\cdot x}||U_n(x)| dx \qquad (\theta \in (0,1)) \\ &\leq C_4 |\xi - \eta|, \end{aligned} \tag{9.5.6}$$

with $C_4 = \dfrac{1}{(2\pi)^{N/2}} C_2 \|x\|_{L^2(K)}$.

Therefore, $(\hat{U}_n)$ satisfies the conditions of Arzela-Ascoli in every ball $\overline{B}_R = \overline{B}(0,R)$, $R > 0$. Then there exists a subsequence of $(\hat{U}_n)$, still denoted by $(\hat{U}_n)$ converging to a certain $\hat{U}_R$ in $C^0(\overline{B}_R)$, for every $R > 0$. It follows that (U_n) is a Cauchy sequence in $H^k(\mathbb{R}^N)$. Indeed, first note that

$$\|u_n - u_{n+m}\|^2_{H^k(\Omega)} \leq \|U_n - U_{n+m}\|^2_{H^k(\mathbb{R}^N)} \tag{9.5.7}$$

$$\begin{aligned} &= \int_{B_R} \langle\xi\rangle^{2k} |\hat{U}_n - \hat{U}_{n+m}|^2 d\xi + \int_{B_R^c} \langle\xi\rangle^{2k} |\hat{U}_n - \hat{U}_{n+m}|^2 d\xi \\ &\leq (1+R^2)^k \|\hat{U}_n - \hat{U}_{n+m}\|^2_{C^0(\overline{B}_R)} \\ &+ \int_{B_R^c} \frac{\langle\xi\rangle^{2k}}{\langle\xi\rangle^{2(k+1)}} \langle\xi\rangle^{2(k+1)} |\hat{U}_n - \hat{U}_{n+m}|^2 d\xi \\ &\leq (1+R^2)^k \|\hat{U}_n - \hat{U}_{n+m}\|^2_{C^0(\overline{B}_R)} + \frac{C_5}{1+R^2} \|\hat{U}_n - \hat{U}_{n+m}\|^2_{H^{k+1}(\mathbb{R}^N)} \\ &\leq C_6 \left((1+R^2)^k \|\hat{U}_n - \hat{U}_{n+m}\|^2_{C^0(\overline{B}_R)} + \frac{1}{1+R^2} \right). \end{aligned} \tag{9.5.8}$$

Next, given $\epsilon > 0$ we choose R such that $\dfrac{C_6}{1+R^2} < \dfrac{\epsilon}{2}$. As $(\hat{U}_n)$ is Cauchy in $C^0(\overline{B}_R)$, there exists n_ϵ such that for all $N > n_\epsilon$ and $m \in \mathbb{N}$ we have $C_6(1+R^2)^k \|\hat{U}_n - \hat{U}_{n+m}\|^2_{C^0(\overline{B}_R)} < \dfrac{\epsilon}{2}$, which shows that (u_n) is a Cauchy sequence in $H^k(\Omega)$, so (u_n) converges in $H^k(\Omega)$. □

We also have the following embedding, which provides an example of how Fourier transform can be successfully used for proving results in a context different from L^2 or H^s spaces.

Theorem 9.5.3 *Let* $s > \dfrac{N}{2} - \dfrac{N}{p}$ *with* $p > 2$*. Then* $H^s \longrightarrow L^p$.

Proof. Let $u \in \mathcal{S}$, $\hat{u} = \mathcal{F}[u]$, and $q \in (1,2)$ the conjugate number of p, $\frac{1}{p} + \frac{1}{q} = 1$. From the continuity of $\mathcal{F}$ from L^q to L^p, see Theorem 9.4.15, and the fact that $u = \mathcal{F}[\hat{u}(-\xi)] = \widehat{\hat{u}(-\xi)}$, it follows $\|u\|_{L^p} = \|\widehat{\hat{u}(-\xi)}\|_{L^p} \le C\|\hat{u}\|_{L^q}$.

If $\|\hat{u}\|_{L^q} \le C\|u\|_{H^s}$ then $\|u\|_{L^p} \le C\|u\|_{H^s}$, and from $\mathcal{S} \overset{d}{\hookrightarrow} H^s$ it follows $H^s \hookrightarrow L^p$. So, to prove, the theorem reduces to proving $\|\hat{u}\|_{L^q} \le C\|u\|_{H^s}$. From Hölder inequality with $\alpha = 2/q$ and $\beta = 1/(1-q/2)$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\hat{u}|^q d\xi &= \int_{\mathbb{R}^N} \langle\xi\rangle^{sq}|\hat{u}|^q \cdot \langle\xi\rangle^{-sq} d\xi \\
&\le \left(\int_{\mathbb{R}^N} \langle\xi\rangle^{2s}|\hat{u}|^2 d\xi\right)^{q/2} \left(\int_{\mathbb{R}^N} \langle\xi\rangle^{\frac{-sq}{1-q/2}} d\xi\right)^{1-q/2} \\
&= \|u\|_{H^s}^q \left(\int_{\mathbb{R}^N} \langle\xi\rangle^{-\frac{2sp}{p-2}} d\xi\right)^{\frac{1}{2}\frac{p-2}{p-1}}.
\end{aligned}$$

But $\int_{\mathbb{R}^N} \langle\xi\rangle^{-\frac{2sp}{p-2}} d\xi < \infty$ iff $N - 2\frac{sp}{p-2} < 0$, which is equivalent to $\frac{s}{N} > \frac{1}{2} - \frac{1}{p}$ and completes the proof. □

As an application of the compactness result in Theorem 9.5.2 and of Lemma 9.5.5 (which is a Functional Analysis result), we have the following result which is useful when studying PDEs of high order.

Corollary 9.5.4 *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set satisfying* $H^k(\Omega) \overset{c}{\hookrightarrow} H^{k-1}(\Omega)$ *and* $a, b > 0$*. Then the following norms in* $H^k(\Omega)$ *are equivalent*

$$\|u\|_{H^k(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^2(\Omega)}, \quad [\![u]\!]_{H^k(\Omega)} = a\|u\|_{L^1(\Omega)} + b|u|_{H^k(\Omega)}. \tag{9.5.9}$$

Proof. As Ω is bounded, by using Hölder inequality it is easy to prove that $[\![u]\!]_{H^k(\Omega)} \le C_1\|u\|_{H^k(\Omega)}$, with a certain $C_1 > 0$. It remains to prove the inverse inequality, i.e. $\|u\|_{H^k(\Omega)} \le C_2[\![u]\!]_{H^k(\Omega)}$, with $C_2 > 0$.

Set $X = H^k(\Omega)$, $Y = H^{k-1}(\Omega)$, and $Z = L^1(\Omega)$. Then X, Y, and Z fulfill the conditions of Lemma 9.5.5. From (9.5.11), for every $\epsilon \in (0,1)$ there exists $C(\epsilon) > 0$ such that

$$\|u\|_{H^{k-1}(\Omega)} \le \epsilon\|u\|_{H^k(\Omega)} + C(\epsilon)\|u\|_{L^1(\Omega)}.$$

This implies

$$\|u\|_{H^k(\Omega)} \leq \frac{1}{1-\epsilon}|u|_{H^k(\Omega)} + \frac{C(\epsilon)}{1-\epsilon}\|u\|_{L^1(\Omega)} \leq C(\epsilon,a,b)[\![u]\!]_{H^k(\Omega)},$$

which proves the lemma.

Lemma 9.5.5 (Ehrling) *Let X, Y, and Z be Banach spaces and assume*

$$X \overset{c}{\hookrightarrow} Y \hookrightarrow Z. \tag{9.5.10}$$

Then for every $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$\forall x \in X, \quad \|x\|_Y \leq \epsilon\|x\|_X + C(\epsilon)\|x\|_Z. \tag{9.5.11}$$

Proof. Assume the lemma fails for a certain $\epsilon_0 > 0$. Then, there exists a sequence x_n in X such that $\|x_n\|_Y > \epsilon_0\|x_n\|_X + n\|x_n\|_Z$. We can assume, without restriction, that $\|x_n\|_X = 1$ (otherwise take $x_n/\|x_n\|_X$ instead of x_n). Then

$$\|x_n\|_Y > \epsilon_0 + n\|x_n\|_Z, \quad \|x_n\|_X = 1. \tag{9.5.12}$$

Since $X \overset{c}{\hookrightarrow} Y$, after passing to a subsequence we may assume that $\lim_{n\to\infty} x_n = y_0$ in Y and so $\|x_n\|_Y$ is bounded. Therefore from (9.5.12), we obtain $\lim_{n\to\infty} x_n = 0$ in Z. From (9.5.10) we have $\|x_n - y_0\|_Z \leq C\|x_n - y_0\|_Y$, so $\lim_{n\to\infty} x_n = y_0$ in Z. The uniqueness of the limit (of x_n in Z) implies $y_0 = 0$, which contradicts (9.5.12) for n large, and so proves the lemma.

9.5.2 Extension and density results in Sobolev spaces

Density and extension results hold in $W^{s,p}(\Omega)$ spaces, but for the sake of simplicity we will focus our analysis only on $W^{k,p}(\Omega)$ spaces. Let us start with two simple results, which are practical in many situations.

Lemma 9.5.6 (Extension by zero) *Let $u \in W_0^{k,p}(\Omega)$, $p \in [1,\infty)$, and $E_0 u = U : \mathbb{R}^N \mapsto \mathbb{R}$, the extension of u in $\mathbb{R}^N$ by zero, i.e. $U(x) = u(x)$ if $x \in \Omega$, $U(x) = 0$ if $x \notin \Omega$. Then $E_0 \in \mathcal{L}(W_0^{k,p}(\Omega); W^{k,p}(\mathbb{R}^N))$.*

Proof. By the definition of $W_0^{k,p}(\Omega)$, there exists (u_n) in $\mathcal{D}(\Omega)$ with $\lim_{n\to\infty} u_n = u$ in $W^{k,p}(\Omega)$. Set $U_n = E_0 u_n$. Then $U_n \in \mathcal{D}(\mathbb{R}^N)$ and $\lim_{n\to\infty} U_n = U$ in $W^{k,p}(\mathbb{R}^N)$, so $U \in W^{k,p}(\mathbb{R}^N)$ and $\|U\|_{W^{k,p}(\mathbb{R}^N)} = \|u\|_{W_0^{k,p}(\Omega)}$.

Lemma 9.5.7 (Truncation and extension by zero) *Let $u \in W^{k,p}(\Omega)$, $p \in [1,\infty]$, $\eta \in \mathcal{D}(\Omega)$. Then $V = E_0(\eta u) \in W^{k,p}(\mathbb{R}^N)$.*

Proof. Indeed, for $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and for any $\alpha \in \mathbb{N}_0^N$, $|\alpha| \le k$, by using (6.1.10) we have

$$\begin{aligned}\langle D^\alpha V, \varphi\rangle &= (-1)^{|\alpha|}\int_{\mathbb{R}^N} V D^\alpha\varphi = (-1)^{|\alpha|}\int_\Omega u\eta D^\alpha\varphi = \int_\Omega D^\alpha(u\eta)\varphi \\ &= \int_\Omega \varphi \sum_{0\le\beta\le\alpha} C_\beta^\alpha D^\beta u D^{\alpha-\beta}\eta \\ &= \int_{\mathbb{R}^N} \varphi E_0\left(\sum_{0\le\beta\le\alpha} C_\beta^\alpha D^\beta u D^{\alpha-\beta}\eta\right).\end{aligned}$$

So $D^\alpha V = E_0\left(\sum_{0\le\beta\le\alpha} C_\beta^\alpha D^\beta u D^{\alpha-\beta}\eta\right) \in L^p(\mathbb{R}^N)$, which proves $V \in W^{k,p}(\mathbb{R}^N)$. $\square$

In the case when Ω is smooth, a function in $W^{k,p}(\Omega)$ can be approximated by functions smooth up to the boundary $\partial\Omega$. The following theorem is instructive, because it highlights the key steps of the method, in particular the step of regularization by convolution, which furthermore shifts the support of the function. The general case is treated by using a partition of unity; see Theorem 9.5.10.

Theorem 9.5.8 *Let $k \in \mathbb{N}$, $p \in [1,\infty)$. Then $\mathcal{D}(\overline{\mathbb{R}}_+^N) \overset{d}{\hookrightarrow} W^{k,p}(\mathbb{R}_+^N)$, where $\mathcal{D}(\overline{\mathbb{R}}_+^N) := \{U|_{\mathbb{R}_+^N},\ \ U \in \mathcal{D}(\mathbb{R}^N)\}$.*

Proof. Let $u \in W^{k,p}(\mathbb{R}_+^N)$. Without loss of generality, we may assume that $\mathrm{supp}(u)$ is bounded. If not, we can always consider $u_n = \eta_n u$, where $\eta_n(x) = \eta(x/n)$ and η is a $\{\overline{B}(0,1), B(0,2)\}$ cut-off function, and show easily that u_n converges to u in $W^{k,p}(\mathbb{R}_+^N)$ similarly as in Theorem 6.1.7.

Now, let (ρ_n) be a mollifier sequence with $\mathrm{supp}(\rho_n) \subset \mathbb{R}_-^N := \{x_N < 0\}$, and consider the sequence $(\check{\rho}_n)$, where in general $\check{\varphi}(x) = \varphi(-x)$. Set $U = E_0(u)$, where E_0 is as in Lemma 9.5.6, $U_n = U * \rho_n$.

We will show that $(u_n) = (U_n|_{\mathbb{R}_+^N})$ is the required sequence. Indeed, as u has bounded support the U also has bounded support, $U_n \in \mathcal{D}$ and furthermore $\lim_{n\to\infty} U_n = U$ in $L^p(\mathbb{R}^N)$, so $\lim_{n\to\infty} u_n = u$ in $L^p(\mathbb{R}_+^N)$. Next we compute $D^\alpha u_n$, $|\alpha| \le k$. For $\varphi \in \mathcal{D}(\mathbb{R}^N) \cap \{\mathrm{supp}(\varphi) \subset \mathbb{R}_+^N\}$, we have

$$\begin{aligned}(-1)^{|\alpha|}\langle D^\alpha u_n, \varphi\rangle_{\mathcal{D}'(\mathbb{R}_+^N)\times\mathcal{D}(\mathbb{R}_+^N)} &= \int_{\mathbb{R}^N} U_n(x)D^\alpha\varphi(x)dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} U(y)\rho_n(x-y)D^\alpha\varphi(x)dydx \\ &= \int_{\mathbb{R}^N} U(y)(\check{\rho}_n * D^\alpha\varphi)(y)dy \\ &= \int_{\mathbb{R}^N} U(y)D^\alpha(\check{\rho}_n * \varphi)(y)dy \\ &= \int_{\mathbb{R}_+^N} u(y)D^\alpha(\check{\rho}_n * \varphi)(y)dy, \qquad (9.5.13)\end{aligned}$$

where the last equality holds for n large, because for such n we have

$$\operatorname{supp}(\check{\rho}_n) \subset \mathbb{R}^N_+, \quad \operatorname{supp}(\check{\rho}_n * \varphi) \subset \sup(\check{\rho}_n) + \operatorname{supp}(\varphi) \subset \mathbb{R}^N_+.$$

So $D^\alpha(\check{\rho}_n * \varphi) \in \mathcal{D}(\mathbb{R}^N_+)$ and then we can integrate by parts in (9.5.13) with no boundary terms, which gives

$$\begin{aligned}
\langle D^\alpha u_n, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} &= \int_{\mathbb{R}^N_+} D^\alpha u(y)(\check{\rho}_n * \varphi)(y)dy \\
&= \int_{\mathbb{R}^N} E_0(D^\alpha u)(y)(\check{\rho}_n * \varphi)(y)dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} E_0(D^\alpha u)(y)\rho_n(x-y)\varphi(x)dxdy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} E_0(D^\alpha u)(y)\rho_n(x-y)\varphi(x)dydx \\
&= \int_{\mathbb{R}^N} (\rho_n * E_0(D^\alpha u))(x)\varphi(x)dx \\
&= \langle \rho_n * E_0(D^\alpha u), \varphi\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)}. \qquad (9.5.14)
\end{aligned}$$

From (9.5.14), we get $D^\alpha u_n = \rho_n * E_0(D^\alpha u)$ and therefore $\lim_{n\to\infty} D^\alpha u_n = D^\alpha u$ in $L^p(\mathbb{R}^N_+)$, which completes the proof. □

The classical construction of $W^{k,p}$ extension operator for $p \in [1,\infty]$ is quite long and technical. We will use Theorem 9.5.8 to construct a $W^{k,p}$ extension operator, $p \geq 1$, with a simple proof.

Theorem 9.5.9 *Let $k \in \mathbb{N}$, $p \in [1,\infty)$. Then there exists a linear continuous extension operator*[10] $E : W^{k,p}(\mathbb{R}^N_+) \mapsto W^{k,p}(\mathbb{R}^N)$.

Proof. Let $E : \mathcal{D}(\overline{\mathbb{R}}^N_+) \mapsto C^k_0(\mathbb{R}^N)$ be defined by

$$(Eu)(x', x_N) := U(x', x_N) = \begin{cases} u(x', x_N), & x_N \geq 0, \\ \displaystyle\sum_{j=0,\dots,k} c_j u\left(x', -\frac{x_N}{j+1}\right), & x_N < 0, \end{cases} \qquad (9.5.15)$$

where $x = (x', x_N)$, $x' = (x_1, \dots, x_{N-1})$, and $u \in \mathcal{D}(\overline{\mathbb{R}}^N_+)$. The coefficients c_j are chosen so that $U \in C^k_0(\mathbb{R}^N)$. For this, it is sufficient to ensure the continuity across $\partial\mathbb{R}^N_+$ of the derivatives $\partial^i_{x_N} U$, $0 \leq i \leq k$. Evaluating these derivatives at $(x', 0)$ leads to

$$\sum_{j=0,\dots,k} \left(-\frac{1}{j+1}\right)^i c_j = 1, \quad i = 0, 1, \dots, k. \qquad (9.5.16)$$

[10] See Theorem 6.3.7 for the general extension result.

The matrix of this system is the so-called "*Vandermonde matrix*", which is nonsingular and so the coefficients c_i are uniquely defined.

E provides the required extension operator. Indeed, clearly E is linear. Furthermore, there exists $C > 0$ such that

$$\|Eu\|_{W^{k,p}(\mathbb{R}^N)} \leq C\|u\|_{W^{k,p}(\mathbb{R}^N_+)}, \qquad \forall u \in \mathcal{D}(\overline{\mathbb{R}}^N_+). \tag{9.5.17}$$

This estimate is proven easily by using simple classical calculus. Therefore as $\mathcal{D}(\overline{\mathbb{R}}^N_+)$ is dense in $W^{k,p}(\mathbb{R}^N_+)$, (9.5.17) implies that E extends to a linear continuous operator from $W^{k,p}(\mathbb{R}^N_+)$ to $W^{k,p}(\mathbb{R}^N)$, still denoted by E, which satisfies (9.5.17) in $W^{k,p}(\mathbb{R}^N_+)$. Finally, $Eu = u$ a.e. in $\mathbb{R}^N_+$ for every $u \in W^{k,p}(\mathbb{R}^N_+)$, because we can choose a sequence (u_n) in $\mathcal{D}(\overline{\mathbb{R}}^N_+)$ with $Eu_n = u_n$ in $\mathbb{R}^N_+$, converging to u in $W^{k,p}(\mathbb{R}^N_+)$ and pointwise in $\mathbb{R}^N_+$.

Theorem 9.5.10 *Assume $\Omega \subset \mathbb{R}^N$ is an open bounded set of class C^k, $k \in \mathbb{N}$, $p \in [1, \infty)$. Then there exists an extension operator*[11] $E = E_\Omega : W^{k,p}(\Omega) \mapsto W^{k,p}(\mathbb{R}^N)$.

Proof. Let $\{(G_i, \theta_i),\ i = 1, \dots, M\}$ be as in Definition 1.2.4, corresponding to some points $\{x_1, \dots, x_M\}$ on $\partial\Omega$ such that $\{G_i,\, i = 1, \dots, M\}$ forms a finite covering of $\partial\Omega$. Let furthermore $\{\eta_i,\, i = 0, 1, \dots, M\}$ be a partition of unity subordinate to $\{G_i,\, i = 1, \dots, M\}$; see Fig. 6.4.1 and Theorem 1.2.2.

For $u \in W^{k,p}(\Omega)$ we have $u = u_0 + u_1 + \cdots + u_M$, with $u_i = u\eta_i$, $i = 0, 1, \dots, M$. Then $u_i \in W^{k,p}(\Omega)$, $i = 0, 1, \dots, M$; see Proposition 6.1.11. We can extend u_0 by zero, namely let $W_0 = E_0 u_0$. As $\mathrm{supp}(u_0) \Subset \Omega$, from Lemma 9.5.7 we get $W_0 \in W^{k,p}(\mathbb{R}^N)$.

We can extend u_i, $i = 1, \dots, M$, to a $W^{k,p}(\mathbb{R}^N)$ function as follows. First define v_i by $v_i = u_i \circ \theta_i$ in Q^+. From Proposition 6.1.13 and the remark following it, we have $v_i \in W^{k,p}(Q^+)$. Note that from $\mathrm{supp}(\eta_i) \Subset G_i \cap \Omega$, we get $\mathrm{supp}(v_i) \subset Q_i \cap Q^+$, with a certain $Q_i \Subset Q$. It is easy to prove that if $\tilde{v}_i$ is the extension by zero of v_i in $\mathbb{R}^N_+$, then $\tilde{v}_i \in W^{k,p}(\mathbb{R}^N_+)$. Next, let $\tilde{V}_i \in W^{k,p}(\mathbb{R}^N)$ be the extension of $\tilde{v}_i$ in $W^{k,p}(\mathbb{R}^N)$ as given by Theorem 9.5.9, and set $w_i = \tilde{V}_i|_Q \circ \theta_i^{-1}$ and $W_i = E_0 w_i$. Clearly $w_i \in W^{k,p}(\mathbb{R}^N_+)$, see Proposition 6.1.13, and as $\mathrm{supp}(w_i) \Subset G_i$, we have $W_i \in W^{k,p}(\mathbb{R}^N)$; see Lemma 9.5.7.

Finally, set $Eu := W_0 + W_1 + \cdots + W_M$. Clearly we have

$$\begin{aligned}
Eu &\in W^{k,p}(\mathbb{R}^N), \\
(Ew_i)(x) &= w_i(x) = u(x)\eta_i(x), \ \ x \in \Omega, \ \ i = 1, \dots, M, \\
(Eu)(x) &= u(x) \sum_{i=0,\dots,M} \eta_i(x) = u(x), \ \ x \in \Omega,
\end{aligned}$$

so Eu is a $W^{k,p}(\mathbb{R}^N)$ extension of u.

Furthermore, E is linear and bounded in $W^{k,p}(\Omega)$, because the maps $u \mapsto u_i \mapsto v_i \mapsto \tilde{v}_i \mapsto \tilde{V}_i \mapsto w_i \mapsto W_i$ are linear and continuous in the appropriate $W^{k,p}$ spaces. □

[11] See Theorem 6.3.7 for the general extension result.

Theorem 9.5.11 *Assume $\Omega \subset \mathbb{R}^N$ satisfies the $W^{k,p}$ extension property, $p \in [1,\infty)$. Then $\mathcal{D}(\overline{\Omega}) \xhookrightarrow{d} W^{k,p}(\Omega)$, where $\mathcal{D}(\overline{\Omega}) := \{U|_\Omega,\ U \in \mathcal{D}(\mathbb{R}^N)\}$.*

Proof. Let $u \in W^{k,p}(\Omega)$ and $U = Eu$ its $W^{k,p}(\mathbb{R}^N)$ extension. From Theorem 6.1.7, there exists a sequence $U_n \in \mathcal{D}(\mathbb{R}^N)$ converging to U in $W^{k,p}(\mathbb{R}^N)$. Set $u_n = U|_\Omega$. Then (u_n) is in $\mathcal{D}(\overline{\Omega})$ and converges to u in $W^{k,p}(\Omega)$.

9.5.3 Boundary traces in Sobolev spaces

The following result shows that the trace operator defined in Theorem 6.4.2 is onto. We give an explicit formula for the right inverse γ^{-1} of γ.

Theorem 9.5.12 *Let $s > \frac{1}{2}$. There exists a bounded linear map $\gamma^{-1} : H^{s-\frac{1}{2}}(\mathbb{R}^{N-1}) \mapsto H^s(\mathbb{R}^N)$ such that $\gamma \circ \gamma^{-1}$ is the identity map in $H^{s-\frac{1}{2}}(\mathbb{R}^{N-1})$.*

Proof. We use the same notations as in Theorem 6.4.2. Let $u \in \mathcal{S}(\mathbb{R}^{N-1})$ and set

$$U(x) = (\gamma^{-1}(u))(x) := \frac{(2\pi)^{1/2}}{I_s}\mathcal{F}^{-1}\left[\frac{\langle\xi'\rangle^{2(s-1/2)}}{\langle\xi\rangle^{2s}}\hat{u}(\xi')\right], \qquad \xi = (\xi', \xi_N),$$

where I_s is defined in Theorem 6.4.2. Note that from the definition of U, we have

$$\hat{U}(\xi) = C_1\frac{\langle\xi'\rangle^{2(s-1/2)}}{\langle\xi\rangle^{2s}}\hat{u}(\xi'), \quad C_1 = \frac{(2\pi)^{1/2}}{I_s}.$$

The map γ^{-1} is well-defined and continuous from $H^s(\mathbb{R}^{N-1})$ to $H^s(\mathbb{R}^N)$ because

$$\begin{aligned}
\|U\|^2_{H^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N}\langle\xi\rangle^{2s}|\hat{U}|^2 d\xi = C_1^2\int_{\mathbb{R}^N}\langle\xi\rangle^{2s}\frac{\langle\xi'\rangle^{4(s-1/2)}}{\langle\xi\rangle^{4s}}|\hat{u}(\xi')|^2 d\xi \\
\textit{(use (6.4.2))} \qquad &= C_1^2\int_{\mathbb{R}^{N-1}}\langle\xi'\rangle^{4(s-1/2)}|\hat{u}(\xi')|^2\int_{\mathbb{R}}\frac{1}{\langle\xi\rangle^{2s}}d\xi_N d\xi' \\
&\le C_2\|u\|^2_{H^{s-1/2}(\mathbb{R}^{N-1})}.
\end{aligned}$$

Finally, we have $U(x',0) = u(x')$ because from (6.4.2) we get

$$\begin{aligned}
U(x',0) &= \frac{1}{(2\pi)^{(N-1)/2}I_s}\int_{\mathbb{R}^{N-1}}e^{i(x'\cdot\xi')}\hat{u}(\xi')\langle\xi'\rangle^{2(s-1/2)}\int_{\mathbb{R}}\frac{1}{\langle\xi\rangle^{2s}}d\xi_N d\xi' \\
&= \frac{1}{(2\pi)^{(N-1)/2}}\int_{\mathbb{R}^{N-1}}e^{i(x'\cdot\xi)'}\hat{u}(\xi')d\xi' = u(x'),
\end{aligned}$$

which together with the inclusion $\mathcal{S}(\mathbb{R}^{N-1}) \hookrightarrow H^{k-1}(\mathbb{R}^{N-1})$ completes the proof of the-orem. □

The following theorem shows that in the case when Ω is a bounded Lipschitz domain, the $W_0^{1,p}(\Omega)$ functions are characterized as $W^{1,p}(\Omega)$ functions with zero boundary trace. The proof demonstrates typical techniques when working with Sobolev spaces, such as truncation, extension, regularization, and shifting/translation of the support of the function.

Corollary 9.5.13 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set and $p \in [1, \infty)$. Then*

$$W_0^{1,p}(\Omega) \;=\; N^{1,p}(\gamma), \text{ where } N^{1,p}(\gamma) = \{u \in W^{1,p}(\Omega),\ \gamma(u) = 0\}. \tag{9.5.18}$$

Proof. We will prove the corollary only in the case $\Omega = \mathbb{R}^N_+$. The proof of the general case is made by transforming the problem to a finite number of problems in $\mathbb{R}^N_+$ by using a partition of unity of $\partial\Omega$.

It is easy to prove $W_0^{1,p}(\Omega) \subset N^{1,p}(\gamma)$. Indeed, if $u \in \mathcal{D}(\Omega)$ then clearly $u \in N^{1,p}(\gamma)$. As $\mathcal{D}(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ and γ is continuous, it follows that $W_0^{1,p}(\Omega) \subset N^{1,p}(\gamma)$.

Now, let $u \in N^{1,p}(\gamma)$ and prove that $u \in W_0^{1,p}(\Omega)$. The proof contains several steps: first a truncation, then an extension by zero and finally an appropriate regularization (which regularizes the function and translates/shifts its support). WLOG we may assume that U has bounded support, as by "*truncation*" (see Theorem 6.1.7) we can always consider a sequence (u_n) with bounded support converging to u in $W^{1,p}(\mathbb{R}^N_+)$. Next, we denote by U the extension of u in $\mathbb{R}^N$ by zero. Then $U \in W^{1,p}(\mathbb{R}^N)$. Indeed, for $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and $|\alpha| = 1$, we have

$$\langle D^\alpha U, \varphi\rangle = -\int_{\mathbb{R}^N} U D^\alpha \varphi dx = -\int_{\mathbb{R}^N_+} u D^\alpha \varphi dx.$$

There exists an open set $\omega \in \mathbb{R}^N_+$ of class C^1 such that $\mathrm{supp}(\varphi) \cap \mathbb{R}^N_+ \subset \omega$. Then, from (6.4.8), for $|\alpha| = 1$ we get

$$\begin{aligned} \int_{\mathbb{R}^N_+} u D^\alpha \varphi dx &= \int_\omega u D^\alpha \varphi dx = \int_{\partial\omega} \gamma(u)\varphi \nu^\alpha d\sigma + \int_\omega D^\alpha u \varphi dx \\ &= \int_{\mathbb{R}^N_+} D^\alpha u \varphi dx. \end{aligned}$$

Hence, $D^\alpha U = E_0(D^\alpha u) \in L^p(\mathbb{R}^N)$, so $U \in W^{1,p}(\mathbb{R}^N)$.

Next, let (ρ_n) be a mollifier sequence with $\mathrm{supp}(\rho_n) \subset B\big((0, \ldots, 0, \frac{1}{n}), \frac{1}{2n}\big)$, and consider $U_n = U * \rho_n$. Clearly $U_n \in C^\infty(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ and $\lim_{n\to\infty} U_n = U$ in $W^{1,p}(\mathbb{R}^N)$. Furthermore, from (5.2.3) we have $\mathrm{supp}(U_n) \subset \overline{\mathrm{supp}(U) + \mathrm{supp}(\rho_n)} \subset \big\{x_N \geq \frac{1}{2n}\big\}$. This proves that $u_n = U_n|_{\mathbb{R}^N_+} \in \mathcal{D}(\mathbb{R}^N_+)$ and $\lim_{n\to\infty} u_n = u$ in $W^{1,p}(\mathbb{R}^N_+)$, so $u \in W_0^{1,p}(\Omega)$.

9.6 Second-order linear elliptic PDEs: weak solutions

9.6.1 Regularity of weak solutions

The method of finding the weak solutions as described in section 7.2 is an elegant way to prove the existence of weak solutions. For example, the problem

$$-\Delta u + u = f \ \ in \ \Omega, \quad u \in H_0^1(\Omega), \quad f \in H^{-1}(\Omega) \tag{9.6.1}$$

has always a unique weak solution $u \in H_0^1(\Omega)$.

If f is more regular than just in $H^{-1}(\Omega)$ and Ω is more regular than just an open set, then one expects that (9.6.1) (or even (7.1.1)) has a weak solution more regular than $H_0^1(\Omega)$. For example, if $\Omega = \mathbb{R}^N$ and $f \in L^2(\mathbb{R}^N)$, by taking Fourier transform of both sides of (9.6.1) we obtain

$$(1+\xi^2)\hat{u} = \hat{f}, \quad \hat{u} = \frac{\hat{f}}{1+\xi^2}, \quad u = \mathcal{F}^{-1}\left[\frac{\hat{f}}{1+\xi^2}\right].$$

Then $u \in H^2(\mathbb{R}^N)$ because $\dfrac{\hat{f}}{1+\xi^2} \in L_2^2(\mathbb{R}^N)$. By analogy, if $f \in L^2(\Omega)$ then one expects that $u \in H_0^1(\Omega) \cap H^2(\Omega)$. We will see that this is true provided Ω is regular.

In this section, we will address the "L^2*-regularity theory*" of the weak solutions to (9.6.1). The results remain the same even for more general second-order elliptic linear PDEs like (7.1.1). As the method of proof is the same, we will focus the analysis on (9.6.1), so as to avoid the technicalities of the general case and highlighting the main ideas.

There is also a "L^p*-regularity theory*" of the weak solutions to second-order elliptic linear PDEs, but the analysis is more difficult. We refer the interested reader to [2, 20, 38] for more in this topic. The result that we will prove in this section is the following.

Theorem 9.6.1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and $u \in W_0^{1,p}(\Omega)$, $p \in (1,\infty)$, be a weak solution to (9.6.1). If Ω is of class C^{m+2} and $f \in W^{m,p}(\Omega)$, $m = 0, 1, \ldots$, then*

$$u \in W^{m+2,p}(\Omega) \quad and \quad \|u\|_{W^{m+2,p}(\Omega)} \le C\|f\|_{W^{m,p}(\Omega)}, \tag{9.6.2}$$

with C independent of f (C depends only on Ω).

The proof of this theorem in the case $p \neq 2$ is more difficult, and we refer the interested reader to the pioneering works on this subject [2, 20, 38]. We will prove the theorem in the case $p = 2$; see, for example, [5].

Note that with $f \in L^2(\Omega)$, the weak solution $u \in H_0^1(\Omega)$ to (9.6.1) satisfies

$$\|u\|_{H^1(\Omega)} \le \|f\|_{L^2(\Omega)}. \tag{9.6.3}$$

The regularity analysis of u will be divided in two parts. First is the regularity in the interior of Ω, which depends only on the data f and not Ω. Next is the regularity near the boundary $\partial\Omega$, which depends on the regularity of the boundary as well. To this end, let $G_1, \ldots, G_n$, $n \in \mathbb{N}$, be a finite covering with open sets of $\partial\Omega$. From Theorem 1.2.2, there exist $\eta_0, \eta_1, \ldots, \eta_n$ in $\mathcal{D}(\mathbb{R}^N)$ and with values in $[0,1]$ such that[12]

$$\begin{array}{l} \sum_{i=0}^n \eta_i = 1 \;\; \textit{in} \;\; \mathbb{R}^N, \\ \operatorname{supp}(\eta_i) \subset G_i \;\; \textit{for all} \;\; i = 1, \ldots, n, \\ \operatorname{supp}(\eta_0) \cap \Sigma_\epsilon = \emptyset \;\; \textit{where} \;\; \Sigma_\epsilon := \{x \in \mathbb{R}^N, \; \textit{dist}(x, \partial\Omega) < \epsilon\}, \end{array}$$

for a certain $\epsilon > 0$. Then we write

$$u = (\eta_0 u) + \sum_{i=1}^n (\eta_i u) =: u_0 + \sum_{i=1}^n u_i. \tag{9.6.4}$$

First, we will prove the regularity of u in the interior of Ω, namely for u_0. The idea is that as $\operatorname{supp}(u_0) \Subset \Omega$, we can use in (7.2.12) as test functions the so-called "*difference quotients*" $D_i^h u_0$, which upon sufficient regularity of f allows one to estimate $\|D_i^h u_0\|_{H^m(\Omega)}$ norms, $m = 1, 2, \ldots$. These estimates imply $H^{m+1}(\Omega)$ regularity of u_0. The regularity of u near the boundary is obtained by first making a change of variable which "*flattens*" the boundary. This allows us to use difference quotients again, which provide similar estimates for u_i functions as for u_0, and ultimately the regularity of u in Ω.

9.6.1.1 Regularity in the interior

We start with some preliminary results. Let $\Omega \subset \mathbb{R}^N$ be an open set and $\{e_1, \ldots, e_N\}$ be the orthogonal standard basis of $\mathbb{R}^N$. For $i = 1, \ldots, N$ and given $h \in \mathbb{R}$ we define $D_i^h u$, the difference quotient of u in direction e_i, by

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}, \quad x \in \Omega \cap \{\textit{dist}(x, \partial\Omega) > |h|\}. \tag{9.6.5}$$

Note that by extending u in $\mathbb{R}^N$ by zero allows to define $D_i^h u$ in $\mathbb{R}^N$.

Lemma 9.6.2 *Assume $\omega \Subset \Omega$, with ω open, and $|h| < \textit{dist}(\omega, \partial\Omega)$. Then for every $u \in L^p(\Omega)$, $p \in [1, \infty)$, and every $v \in C^0(\Omega)$ bounded with* $\operatorname{supp}(v) \subset \omega$, *we have*

$$D_i^h(uv)(x) \;=\; u(x) D_i^h v(x) + v(x + he_i) D_i^h u(x), \;\; x \in \omega, \tag{9.6.6}$$

$$\int_\Omega D_i^h u(x) v(x) \;=\; \int_\Omega u(x) D_i^{-h} v(x) dx. \tag{9.6.7}$$

[12]Given $A, B \subset \mathbb{R}^N$, $\textit{dist}(A, B) = \inf\{|a - b|, \; a \in A, \; b \in B\}$.

Proof. Indeed, we get easily

$$D_i^h(uv)(x) = u(x)D_i^h v(x) + v(x+he_i)D_i^h u(x),$$

and by changing the variable we obtain

$$\int_\Omega D_i^h u(x)v(x)dx = \int_\Omega \frac{u(x+he_i)-u(x)}{h}v(x)dx = \int_\Omega u(x)\frac{v(x-he_i)-v(x)}{h}dx,$$

which proves the lemma.

Lemma 9.6.3 *Let $u \in W^{1,p}(\Omega)$, $p \in [1,\infty)$. Then*

$$\|D_i^h u\|_{L^p(\omega)} \le \|\partial_i u\|_{L^p(\Omega)}, \quad \forall \omega \Subset \Omega, \quad \omega \ \ open, \ \ |h| < dist(\omega, \partial\Omega). \tag{9.6.8}$$

Proof. Let $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$. For $t \in \mathbb{R}$ set $v(t) = u(x+the_i)$. Then $v \in C^\infty(\mathbb{R})$, $v'(t) = he_i \cdot \nabla u(x+the_i)$, and

$$D_i^h u(x) = \frac{v(1)-v(0)}{h} = \frac{1}{h}\int_0^1 v'(t)dt = \int_0^1 e_i \cdot \nabla u(x+the_i)dt.$$

Therefore, it follows

$$\begin{aligned} \|D_i^h u\|_{L^p(\omega)}^p &= \int_\omega \left|\int_0^1 |e_i \cdot \nabla u(x+the_i)dt\right|^p dx \\ &\le \int_0^1 \int_\omega |\nabla u(x+the_i)|^p dxdt \le \int_0^1 \int_\Omega |\nabla u(y)|^p dydt \\ &= \|\nabla u\|_{L^p(\Omega)}^p, \end{aligned}$$

because from $\omega \Subset \Omega$, $|h| < d(\omega, \partial\Omega)$ we get $y = x + the_i \in \Omega$. This inequality remains true for all $u \in W^{1,p}(\Omega)$; see Theorem 6.1.8.

Lemma 9.6.4 *Let $p \in (1,\infty)$ and $u \in L^p(\Omega)$. Assume that there exists $C > 0$ such that*

$$\|D_i^h u\|_{L^p(\omega)} \le C, \quad \forall \omega \Subset \Omega, \quad \omega \ \ open, \quad |h| < dist(\omega, \partial\Omega), \ i = 1, \dots, N. \tag{9.6.9}$$

Then $u \in W^{1,p}(\Omega)$ and

$$\|\partial_i u\|_{L^p(\Omega)} \le C. \tag{9.6.10}$$

Proof. Let $\varphi \in \mathcal{D}(\Omega)$. Then

$$\langle \partial_i u, \varphi \rangle = -\int_\Omega u(x)\partial_i\varphi(x)dx = -\lim_{h\to 0}\int_\Omega u(x)D_i^h\varphi(x)dx = \lim_{h\to 0}\langle D_i^h u, \varphi\rangle, \tag{9.6.11}$$

because $\varphi \in \mathscr{D}(\Omega)$. As $\|D_i^h u\|_{L^p(\omega)}$ is bounded, there exists a sequence (h) tending to zero and $v \in L^p(\omega)$, such that $\lim_{h\to 0} D_i^h u = v$ weakly in $L^p(\omega)$, see Theorem 1.3.8, i.e.

$$\lim_{h\to 0}\langle D_i^h u, \varphi\rangle = \int_\Omega v\varphi, \quad \forall \varphi \in \mathscr{D}(\omega). \tag{9.6.12}$$

From the arbitrariness of ω, we have $v \in L^p(\Omega)$. Moreover, from (9.6.11) and (9.6.12) it follows

$$\langle \partial_i u, \varphi\rangle = \int_\Omega v\varphi dx, \quad \forall \varphi \in \mathscr{D}(\Omega),$$

which proves that $\partial_i u \in L^p(\Omega)$. Finally, from (9.6.11) we obtain

$$\left|\int_\Omega \partial_i u\varphi dx\right| \le C\|\varphi\|_{L^{p'}(\Omega)}, \quad \frac{1}{p}+\frac{1}{p'} = 1.$$

As $\mathscr{D}(\Omega)$ is dense in $L^{p'}(\Omega)$, we can take $\varphi = \partial_i u|\partial_i u|^{p-2} \in L^{p'}(\Omega)$ and then it follows $\int_\Omega |\partial_i u|^p dx \le C\left(\int_\Omega |\partial_i u|^p dx\right)^{1/p'}$, so $\|\partial_i u\|^p_{L^p(\Omega)} \le C\|\partial u\|^{p/p'}_{L^p(\Omega)}$ and this proves (9.6.10). □

We note that the advantage of using $D_i^h u$, $i = 1,\dots,N$, is that they represent a constructive method for proving further regularity of u. For example, Lemma 9.6.4 asserts H^1 regularity for u only from a condition for $D_i^h u$. We will use this technique in the following two theorems.

Theorem 9.6.5 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume $u \in H_0^1(\Omega)$ is a weak solution to (9.6.1). If $f \in H^m(\Omega)$ then u_0 defined in (9.6.4) satisfies*

$$u_0 \in H^{m+2}(\Omega), \quad \|u_0\|_{H^{m+2}(\Omega)} \le C\|f\|_{H^m(\Omega)}, \tag{9.6.13}$$

with C independent of f (C depends only on Ω).

Proof. Let us first deal with the case $m = 0$, so $f \in L^2(\Omega)$. Denote still by u_0 the extension of u_0 in $\mathbb{R}^N$ by zero. So $u_0 \in H^1(\mathbb{R}^N)$, see Lemma 9.5.7, and from (9.6.4) and (9.6.1) it follows that u_0 is a weak solution in $\mathbb{R}^N$ to

$$-\Delta u_0 + u_0 = g := \eta_0 f - 2(\nabla\eta_0\cdot\nabla u) - (\Delta\eta_0)u. \tag{9.6.14}$$

Note that $g \in L^2(\mathbb{R}^N)$. For $\varphi \in H^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \nabla u_0\cdot\nabla\varphi + u_0\varphi = \int_{\mathbb{R}^N} g\varphi. \tag{9.6.15}$$

Let $i \in \{1,\dots,N\}$, $h > 0$, and take $\varphi = D^{-h}D^h u_0 := D_i^{-h}D_i^h u_0$. Note that $\varphi \in H^1(\mathbb{R}^N)$ as a linear combination of $H_0^1(\mathbb{R}^N)$ functions. From (9.6.7), the fact that ∂_j and D^h commute, and (9.6.15), we obtain

$$\int_{\mathbb{R}^N} |\nabla D^h u_0|^2 + \int_{\mathbb{R}^N} |D^h u_0|^2 = \int_{\mathbb{R}^N} gD^{-h}D^h u_0.$$

This gives $\|D^h u_0\|^2_{H^1(\mathbb{R}^N)} \leq \|g\|_{L^2(\mathbb{R}^N)} \|D^{-h} D^h u_0\|_{L^2(\mathbb{R}^N)}$. But from Lemma 9.6.3, we have $\|D^{-h}(D^h u_0)\|_{L^2(\mathbb{R}^N)} \leq \|D^h u_0\|_{H^1(\mathbb{R}^N)}$. Combining the two last inequalities gives

$$\|D^h u_0\|_{H^1(\mathbb{R}^N)} \leq \|g\|_{L^2(\mathbb{R}^N)}. \tag{9.6.16}$$

Thanks to Lemma 9.6.4, this implies $u_0 \in H^2(\mathbb{R}^N)$, so $u_0 \in H^2(\Omega)$. Furthermore, from (9.6.16), (9.6.14), and (9.6.3), we obtain

$$\begin{aligned}\|\partial_i u_0\|_{H^1(\Omega)} &\leq \|g\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\Omega)} + 2\|\eta_0\|_{H^2(\Omega)} \|u\|_{H^1(\Omega)} \\ &\leq (1 + 2\|\eta_0\|_{H^2(\Omega)})\|f\|_{L^2(\Omega)},\end{aligned}$$

which proves the theorem in the case $m = 0$.

For $m \geq 1$, we proceed by recurrence w.r.t. m. So, assume that u satisfies Theorem 9.6.5 up to $m-1$ and let us prove it for m. This means that $u_0 \in H^{m+1}(\mathbb{R}^N)$, $g \in H^m(\Omega)$, and to prove is $u_0 \in H^{m+2}(\mathbb{R}^N)$. Let $\alpha \in \mathbb{N}_0^N$, $|\alpha| = m$. Then $D^\alpha u_0 \in H^1(\mathbb{R}^N)$, $D^\alpha g \in L^2(\mathbb{R}^N)$, and

$$\begin{aligned}-\Delta(D^\alpha u_0) + D^\alpha u_0 &= D^\alpha g, \quad \textit{weakly, i.e.} \\ \int_{\mathbb{R}^N} \nabla(D^\alpha u_0)\cdot\nabla\varphi + (D^\alpha u_0)\varphi &= \int_{\mathbb{R}^N} D^\alpha g \varphi, \quad \forall \varphi \in H^1(\mathbb{R}^N).\end{aligned}$$

This is obtained from (9.6.15) by taking $D^\alpha\varphi$ as a test function and integrating by parts. Then as in the case $m = 0$, we obtain $D^\alpha u_0 \in H^2(\mathbb{R}^N)$, so $u_0 \in H^{m+2}(\Omega)$, and the estimation (9.6.13) follows.

9.6.1.2 Regularity near the boundary

Now we will prove that $u_i := \eta_i u \in H^{m+2}(\Omega)$, for all $i = 1, \ldots, N$. The proof strongly uses the regularity of Ω and of f. Note that $u_i \in H^1_0(G_i^+)$, where $G_i^+ = G_i \cap \Omega$. Indeed, if $\varphi_k \in \mathscr{D}(\Omega)$ such that $\lim_{k\to\infty} \varphi_k = u_i$ in $H^1(\Omega)$, then $\eta_i\varphi_k \in \mathscr{D}(G_i^+)$ and $\lim_{k\to\infty}(\varphi_k\eta_i) = u_i$ in $H^1(G_i^+)$. Furthermore, u_i satisfies

$$-\Delta u_i = g_i := \eta_i f - \eta_i u - 2(\nabla\eta_i\cdot\nabla u) - (\Delta\eta_i)u \quad \textit{in} \ G_i^+. \tag{9.6.17}$$

From (9.6.17) and (9.6.3), we have $g_i \in L^2(G_i^+)$ and

$$\int_{G_i^+} (\nabla u_i \cdot \nabla\psi)dx = \int_{G_i^+} g_i\psi dx, \quad \forall\psi \in H^1_0(G_i^+), \tag{9.6.18}$$

$$\|u_i\|_{H^1(G_i^+)} \leq C\|f\|_{L^2(\Omega)}, \tag{9.6.19}$$

with C depending only on Ω.

We will prove the regularity of u_i by using the method of translations. However, as the equation (9.6.18) is in G_i^+, we cannot take $\varphi = D_i^{-h} D_i^h u_i$, because this φ may not be in $H^1_0(G_i^+)$. For this purpose, we will change the variable and write (9.6.18) in Q^+, see section 1.2, which allows the use of the method of translations.

More precisely, as Ω is of class C^{m+2} there exists $\theta_i \in C^{m+2}(\overline{Q};\overline{G}_i)$, invertible with inverse $\zeta_i \in C^{m+2}(\overline{G}_i;\overline{Q})$ such that (see Fig 6.4.1)

$$\theta_i(Q^+) = G_i^+, \qquad \theta_i(Q^0) = G_i^0, \quad G_i^0 := G_i^+ \cap \partial\Omega.$$

We change the variable and write (9.6.18) in terms of $v_i \in H_0^1(Q^+)$. Namely, we set

$$x = \theta_i(y) \in G, \quad y = \zeta_i(x) \in Q, \quad v_i(y) = u_i(\theta_i(y)), \quad u_i(x) = v_i(\zeta_i(x)). \tag{9.6.20}$$

The strategy for $m = 0$ is as follows. After changing the variable, it turns out that v_i satisfies a certain PDE in Q^+ (see (9.6.18)). In Q^+, we can make translations $D^h := D_j^h$, $j = 1, \dots, N-1$, and by using the technique of the regularity in the interior, we obtain $\partial_{ij}^2 v_i \in L^2(Q^+)$, $j = 1, \dots, N-1$, $i = 1, \dots, N$. The regularity $\partial_{NN}^2 v_i \in L^2(Q^+)$ is obtained by using the weak form equation for v_i, which concludes that $v_i \in H^2(Q^+)$, and therefore from Proposition 6.1.13 we obtain $u_i \in H^2(G_i^+)$. For $m > 0$ we will proceed by recurrence.

The following lemma is important. It shows that under the change of variables (9.6.20), the equation (9.6.18) is transformed to other *elliptic* PDEs.

Lemma 9.6.6 *Under the conditions of Theorem 9.6.1, $v_i \in H_0^1(Q^+)$ and it solves*

$$\int_{Q^+} \sum_{k,l=1}^{N} a_{kl}(y)(\partial_k v_i(y)\partial_l \varphi(y))dy = \int_{Q^+} b_i(y)\varphi(y)dy, \quad \forall \varphi \in H_0^1(Q^+), \tag{9.6.21}$$

where $a_{kl} \in C^{m+1}(\overline{Q^+})$, $b_i \in H^m(Q^+)$ and satisfy

$$a_{kl}(y) = |\mathrm{Jac}(\theta_i)|([\nabla\theta_i]^{-1} \cdot {}^t[\nabla\theta_i]^{-1})_{kl}, \quad a_0|\xi|^2 \le \sum_{k,l=1}^{n} a_{kl}\xi_k\xi_l \le A_0|\xi|^2, \tag{9.6.22}$$

$$b_i(y) = |\mathrm{Jac}(\theta_i)| g_i(\theta_i(y)), \tag{9.6.23}$$

with $A_0, a_0 > 0$.

Proof. For arbitrary $\varphi \in H_0^1(Q^+)$, set $\psi(y) = \varphi(x)$, $x = \theta_i(y)$. Then $\psi \in H_0^1(G_i^+)$ and

$$\begin{aligned}\nabla u_i(x) \cdot \nabla\psi(x) &= {}^t({}^t[\nabla\theta_i(y)]^{-1} \cdot \nabla v_i(y)]) \cdot ({}^t[\nabla\theta_i(y)]^{-1} \cdot \nabla\varphi(y)) \\ &= \nabla v_i(y) \cdot [\nabla\theta_i(y)]^{-1} \cdot {}^t[\nabla(y)]^{-1} \cdot \nabla\varphi(y).\end{aligned}$$

Therefore, changing the variable in (9.6.18) gives

$$\begin{aligned}\int_{Q^+} b_i(y)\varphi(y)dy &= \int_{G_i^+} g_i(x)\psi(x)dx \\ &= \int_{G_i^+} (\nabla u_i(x) \cdot \nabla\psi(x))dx\end{aligned}$$

$$\begin{aligned} &= \int_{G_i^+} \nabla v_i(y) \cdot [\nabla\theta_i(y)]^{-1} \cdot {}^t[\nabla\theta_i(y)]^{-1} \cdot \nabla\varphi(y) |\mathrm{Jac}(\theta_i(y))| dy \\ &= \int_{Q^+} \sum_{k,l=1}^{N} a_{kl}(y)(\partial_k v_i(y) \partial_l \varphi(y)) dy. \end{aligned}$$

For arbitrary $\xi = (\xi_1, \cdots, \xi_N)$ and $y \in Q^+$, we have

$$A_0|\xi|^2 \geq \sum_{k,l=1}^{N} a_{kl}\xi_k\xi_l = |{}^t[\nabla\theta_i(y)]^{-1} \cdot \xi|^2 |\mathrm{Jac}(\theta_i(y))| \geq a_0|\xi|^2,$$

for certain $A_0, a_0 > 0$, because $a_{kl} \in C^{m+1}(\overline{Q})$, $[\nabla\theta_i]$ is not singular in $\overline{Q^+}$ and $|{}^t[\nabla\theta_i(y)]^{-1} \cdot \xi|$ is a norm in $\mathbb{R}^N$ equivalent to $|\xi|$ (independently of y). □

Now we can prove this theorem.

Theorem 9.6.7 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class C^{m+2} and assume $u \in H_0^1(\Omega)$ is a weak solution to (9.6.1). If $f \in H^m(\Omega)$ then $u_i \in H^{m+2}(\Omega)$, $i = 1, \ldots, n$, and the following estimation holds:*

$$\|u_i\|_{H^{m+2}(\Omega)} \leq C\|f\|_{H^m(\Omega)}, \tag{9.6.24}$$

with C independent of f (C depends only on Ω)

Proof. Let us first deal with the case $m = 0$, so $f \in L^2(\Omega)$. From Proposition 6.1.13, it is enough to prove that v_i solving (9.6.21) satisfies $v_i \in H^2(Q^+)$. For $j = 1, \ldots, N-1$, we consider $|h|$ small enough such that $\varphi = D_j^{-h}D_j^h v_i \in H_0^1(Q^+)$ (this is possible because $\mathrm{sup}(u_i) \subset G_i^+$). Replacing φ in (9.6.21) and using (9.6.7) gives

$$\int_{Q^+} \sum_{k,l=1}^{N} D_j^h \left(a_{kl}(y)\partial_k v_i(y)\right) D_j^h(\partial_l v_i(y)) dy = \int_{Q^+} b_i(y) D_j^{-h} D_j^h v_i(y) dy,$$

which in combination with (9.6.6) implies

$$\begin{aligned} \int_{Q^+} \sum_{k,l=1}^{N} a_{kl}(y + he_j)(D_j^h \partial_k v_i)(D_j^h \partial_l v_i) dy &= \int_{Q^+} b_i D_j^{-h} D_j^h v_i dy \\ &- \int_{Q^+} \sum_{k,l=1}^{N} (D_j^h a_{kl})(\partial_k v_i (D_j^h \partial_l v_i)) dy. \end{aligned}$$

Using the fact that a_{kl} are C^1 functions, (9.6.22) and Lemma 9.6.3, from the last equation we obtain

$$a_0\|D_j^h \nabla v_i\|^2_{L^2(Q^+)} \leq \left(\|b_i\|_{L^2(Q^+)} + \|a_{kl}\|_{C^1(\overline{Q^+})}\|\nabla v_i\|_{L^2(Q^+)}\right) \|D_j^h \nabla v_i\|_{L^2(Q^+)}.$$

Note that from b_i in (9.6.23), g_i in (9.6.17), and the estimations (9.6.19), (9.6.3), we obtain

$$\|b_i\|_{L^2(Q^+)} + \|a_{kl}\|_{C^1(\overline{Q^+})}\|\nabla v_i\|_{L^2(Q^+)} \leq C\|f\|_{L^2(\Omega)}.$$

Therefore, the last two inequalities and Lemma 9.6.3 yield

$$\|\partial^2_{jk} v_i\|_{L^2(Q^+)} \leq C\|f\|_{L^2(\Omega)}, \quad j = 1, \dots, N-1,\ k = 1, \dots, N, \tag{9.6.25}$$

with C depending only on Ω. Note that this inequality implies that for all $\varphi \in H^1_0(Q^+)$, we have

$$\left|\int_{Q^+} \partial_j v_i \partial_k \varphi dy\right| \leq C\|f\|_{L^2(\Omega)}\|\varphi\|_{L^2(Q^+)}, \quad i = 1, \dots, N-1,\ j = 1, \dots, N. \tag{9.6.26}$$

From (9.6.25), to prove $v_i \in H^2(Q^+)$ is enough to show that $\partial_{NN} v_i \in L^2(Q^+)$. From $\langle \partial_{NN} v_i, \varphi\rangle_{\mathcal{D}'(Q^+)\times\mathcal{D}(Q^+)} = -\int_{Q^+} \partial_N v_i \partial_N \varphi dy$ and from Riesz representation Theorem 7.2.1 in $L^2(\Omega)$, if

$$\left|\int_{Q^+} \partial_N v_i \partial_N \varphi dy\right| \leq C\|\varphi\|_{L^2(Q^+)}, \quad \forall\varphi \in \mathcal{D}(Q^+), \tag{9.6.27}$$

then $\int_{Q^+} \partial_N v_i \partial_N \varphi dy = (z, \varphi)_{L^2(\Omega)}$ with a certain $z \in L^2(\Omega)$, and so $\partial_{NN} v_i = z \in L^2(Q^+)$ and $\|z\|_{L^2(\Omega)} \leq C$. Taking $\dfrac{\varphi}{a_{NN}}$ instead of φ in (9.6.21) (note that $a_{NN} \geq a_0$) gives

$$\begin{aligned}\int_{Q^+} \partial_N v_i \partial_N \varphi dy &= \int_{Q^+} \frac{\varphi}{a_{NN}} b_i + \int_{Q^+} \frac{\varphi}{a_{NN}} \partial_N a_{NN} \partial_N v_i \\ &+ \int_{Q^+} \sum_{(k,l)\neq(N,N)} \left(\frac{\varphi}{a_{NN}} \partial_k v_i \partial_l a_{kl} - \partial_l\left(\frac{a_{kl}}{a_{NN}}\varphi\right)\partial_k v_i\right).\end{aligned} \tag{9.6.28}$$

Combining (9.6.28) and (9.6.26) gives (9.6.27) with $C\|f\|_{L^2(\Omega)}$ instead of C, and this proves $v_i \in H^2(Q^+)$.

For $m \geq 1$, we proceed by recurrence as in Theorem 9.6.5. So let us assume that the theorem holds for up to $m-1$ and prove it for m. As in Theorem 9.6.5, $D^\alpha v_i$, $|\alpha| = m$ solves

$$\int_{Q^+} \sum_{k,l=1}^{N} a_{kl}(\partial_k D^\alpha v_i)(\partial_l \varphi) dy = \int_{Q^+} \left((D^\alpha b_i)\varphi - \sum_{k,l=1}^{N} (D^\alpha a_{kl})(\partial_k v_i \partial_l \varphi) dy\right) dy, \tag{9.6.29}$$

for all $\varphi \in H^1_0(Q^+)$. This equation is obtained from (9.6.21) by taking $D^\alpha\varphi$ instead of φ, with $\varphi \in \mathcal{D}(Q^+)$, integrating by parts, and then proceeding by density. Note that $D^\alpha b_i \in L^2(Q^+)$, $D^\alpha a_{kl} \in C^1(\overline{Q^+})$. Then we proceed with (9.6.29) as with (9.6.21) and then $D^\alpha v_i \in H^2(Q^+)$, so $v_i \in H^{m+2}(Q^+)$.

Finally, the estimation (9.6.24) follows (9.6.25) and (9.6.27), which completes the proof of the theorem.

Proof of Theorem 9.6.1 We consider only the case $p = 2$. We have $u = u_0 + \sum_{i=1}^{n} u_i$. From Theorem 9.6.5, we have $u_0 \in H^{m+2}(\Omega)$, and from Theorem 9.6.7 we have $u_i \in H^{m+2}(\Omega)$, so $u \in H^{m+2}(\Omega)$. The estimation (9.6.2) follows from (9.6.13) and (9.6.24).

Bibliography

[1] R. Adams and J. Fournier. *Sobolev Spaces. Second edition.* Elsevier, 2003.

[2] S. Agmon, A. Douglis, and L. Nirenberg. “Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions”. In: *Comm. Pure Appl. Math.* I, II (1959).

[3] H. Amann. *Vector-valued distributions of Fourier transform multipliers*. University of Zürich. 2003. url: https://www.math.uzh.ch/amann/files/distributions.ps.

[4] S. Benzoni-Gavage and D. Serre. *Multi-dimensional Hyperbolic Partial Differential Equations: First-order Systems and Applications*. Oxford Mathematical Monographs. Oxford University Press, 2007.

[5] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011.

[6] C. Caratheodory. *Calculus of Variations and Partial Differential Equations. Part I: Partial Differential Equations of the First Order*. San Francisco, London, Amsterdam: Holden-Day, Inc., 1965.

[7] H. Cartan. *Differential calculus*. Paris: Hermann, 1971.

[8] Ph. Ciarlet. *Linear and Nonlinear Functional Analysis with Applications*. SIAM, 2013.

[9] R. Courant and D. Hilbert. *Methods of Mathematical Physics*. Vol. 2. Wiley-Interscience, 1989.

[10] C. M. Dafermos. *Hyperbolic Conservation Laws in Continuum Physics*. Springer, 2010.

[11] R. Dautray and J.-L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology. Evolution Problems*. Vol. V. Berlin: Springer-Verlag, 2000.

[12] R. Dautray and J.-L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology. Functional and Variational Methods*. Vol. II. Berlin: Springer-Verlag, 2000.

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6

[13] R. Dautray and J.-L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology. Spectral Theory and Applications*. Vol. III. Berlin: Springer-Verlag, 2000.

[14] F. Demengel and G. Demengel. *Functional Spaces for the Theory of Elliptic Partial Differential Equations. Fractional Sobolev Spaces*. Springer, 2012.

[15] J. Diestel and Jr. J. J. Uhl. *Vector Measures. Mathematical Surveys and Monographs*. Vol. 15. AMS, 1977.

[16] L. C. Evans. *Partial Differential Equations*. Vol. 19. Graduate Studies in Mathematics. AMS, 1998.

[17] E. Fabes, O. Mendez, and M. Mitrea. "Boundary Layers on Sobolev-Besov Spaces and Poisson's Equation for the Laplacian in Lipschitz Domains". In: *Journal of Functional Analysis* 159 (1998), pp. 323–368.

[18] G. G. Friedlander and M. Joshi. *Introduction to the theory of distributions*. Cambridge University Press, 1999.

[19] P. Garabedian. *Partial Differential Equations*. Wiley, 1964.

[20] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Secondorder*. Vol. 224. Springer, 2001.

[21] Ch. Heil. *Introduction to Real Analysis*. Graduate Texts in Mathematics. Springer, 2019.

[22] L. Hörmander. *The analysis of linear partial differential operators. Distribution Theory and Fourier Analysis*. Vol. I. Springer-Verlag, 1983.

[23] G. Hörmann and R. Steinbauer. *Lecture notes on Theory of distributions*. Fakultät für Mathematik, Universität Wien, 2009. url: https://www.mat.univie.ac.at/ stein/lehre/SoSem09/distrvo.pdf.

[24] A. Jonsson and H. Wallin. *Function Spaces on Subsets of* $\mathbb{R}^n$. New York: Harwood Academic, 1984.

[25] D. Kim. "Trace theorems for Sobolev-Slobodeckij spaces with or without weights". In: *J. Funct. Spaces Appl.* 5.3 (2007), pp. 243–268.

[26] E. Kreyszig. *Introductory functional analysis with applications*. New York: Wiley, 1978.

[27] O. A. Ladyzhenskaja, V. A. Solonnikov, and Ural'ceva N. N. *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs. Providence, RI: AMS, 1968.

[28] S. Lang. *Real and Functional Analysis*. Graduate Texts in Mathematics, Third edition. Springer, 1991.

[29] P. Lax. *Functional Analysis*. Pure and Applied Mathematics, A Wiley-Interscience Series of Texts, Monographs and Tracts. Wiley, 2002.

[30] P. Lax. *Hyberbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. SIAM, 1973.

[31] P. Lax. "Nonlinear hyperbolic equations". In: *Communications on Pure and Applied Mathematics* VI (1953), pp. 231-258.

[32] P. G. Leoch. *Hyperbolic systems of conservation laws. The theory of classical and nonclassical shock waves*. Lectures in Mathematics ETH Zürich. Basel: Birkhäuser Verlag, 2002.

[33] G. Lefloni. *A First Course in Sobolev Spaces*. Graduate Studies in Mathematics. AMS, 2009.

[34] E. H. Lieb and M. Loss. *Analysis*. AMS, 2001.

[35] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications*. Vol. 1. Springer-Verlag, 1972.

[36] J. Málek et al. *Weak and Measure-valued Solutions to Evolutionary PDEs*. CRC Press, 1996.

[37] J. Marschall. "The trace of Sobolev-Slobodeckij spaces on Lipschitz domains". In: *Manuscripta Math.* 58 (1987), pp. 47–65.

[38] N. G. Meyers. "An L^p estimate for the gradient of solutions of second-order elliptic divergence equations". In: *Annali della Scuola Normale Superiore di Pisa*. Classe di Scienze 3e série 17.3 (1963), pp. 189–206.

[39] D. Mitrea, M. Mitrea, and L. Yan. "Boundary value problems for the Laplacian in convex and semiconvex domains". In: *Journal of Functional Analysis* 258 (2010), pp. 2507–2585.

[40] J. Neças. *Direct Methods in the Theory of Elliptic Equations*. Springer, 2012.

[41] E. Di Nezza, G. Palatucci, and E. Valdinoci. "Hitchhiker's guide to the fractional Sobolev spaces". In: *Bulletin des Sciences Mathématiques* 136 (5 2012), pp. 521-573.

[42] L. Nirenberg. *Topics in Nonlinear Functional Analysis*. Vol. 6. AMS and Courant Institute of Mathematical Sciences, 2001.

[43] M. Renardy and R. Rogers. *An Introduction to Partial Differential Equations, Second Edition*. Springer, 2003.

[44] Th. Runst and W. Sickel. *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*. Vol. 3. De Gruyter Series in Nonlinear Analysis and Applications. De Gruyter, 1996.

[45] V. S. Rychkov. "On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains". In: *J. Lond. Math. Soc.* 60 (1 1999), pp. 237–257.

[46] C. Schneider. "Traces of Besov and Triebel-Lizorkin spaces on domains". In: *Math. Nachr.* 284.5-6 (2011), pp. 572–586.

[47] L. Schwartz. *Théorie des distributions*. Hermann, 1978.

[48] E. M. Stein. *Singular Integrals and Differentiability of Functions*. Vol. 30. Princeton Mathematical Series. Princeton University Press, 1971.

[49] E. M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Vol. 32. Princeton Mathematical Series. Princeton University Press, 1990.

[50] R. S. Strichartz. *A Guide to Distribution Theory and Fourier Transforms*. World Scientific Publishing, 2003.

[51] L. Tartar. *An introduction to Sobolev spaces and interpolation spaces*. Springer, 2007.

[52] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*. Vol. 343. AMS Chelsea Publishing, 1984.

[53] F. Treves. *Basic Partial Differential Equations*. Academic Press, 1975.

[54] H. Triebel. *Interpolation Theory, Function Spaces and Differential Operators*. Amsterdam, New York, Oxford: North-Holland, 1978. 214

Index

A. Novruzi, *A Short Introduction to Partial Differential Equations*, CMS/CAIMS Books in Mathematics 11,
https://doi.org/10.1007/978-3-031-39524-6